Section 1.1 A Preview of Calculus

- **Understand what calculus is and how it compares with precalculus.**
- **Understand that the tangent line problem is basic to calculus.**
- **Understand that the area problem is also basic to calculus.**

STUDY TIP As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as "The area of the region is 4.6 square meters."
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

What Is Calculus?

Calculus is the mathematics of change—velocities and accelerations. Calculus is also the mathematics of tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- A tangent line to a circle can be analyzed with precalculus mathematics. To analyze a tangent line to a general graph, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process. So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages. The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle. The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

Precalculus mathematics → Limit process → Calculus

Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

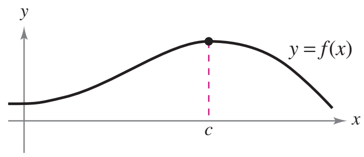
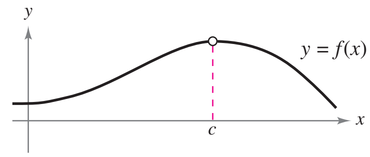
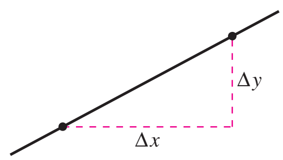
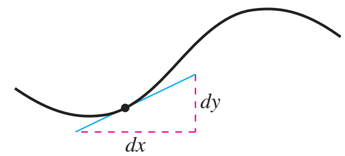
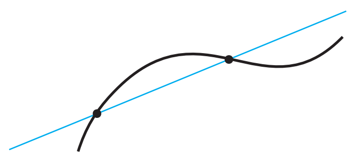
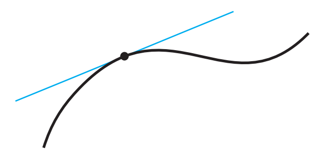
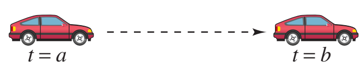
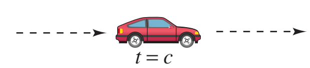
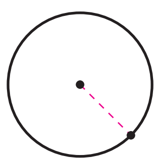
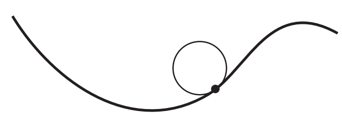
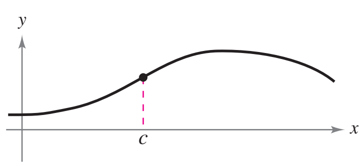
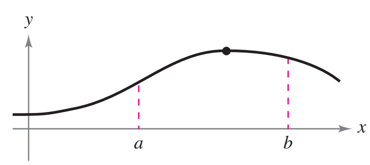
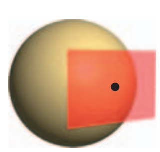
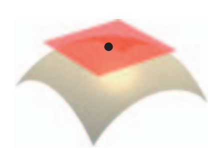
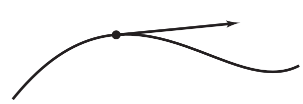
On the following two pages some familiar precalculus concepts coupled with their calculus counterparts are listed. Throughout the text, your goal should be to learn how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don't worry if you are unfamiliar with some of the "old formulas" listed on the following two pages—you will be reviewing all of them.

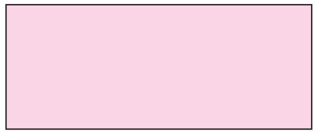
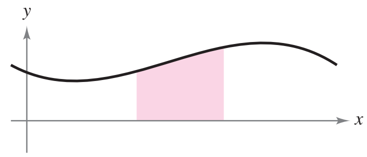
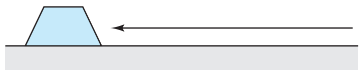
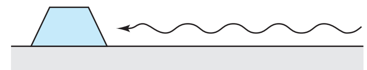
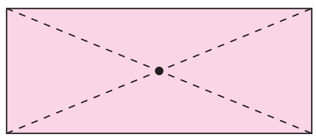
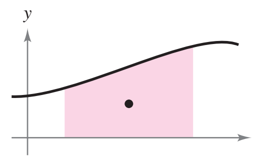
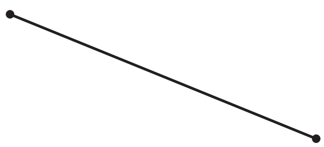
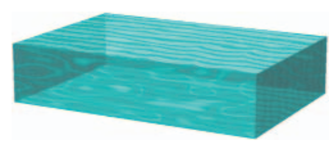
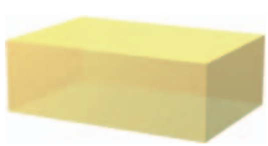
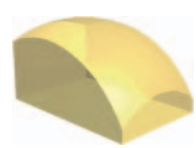
As you proceed through this text, come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as shown.

Chapter P: Preparation for Calculus	Precalculus
Chapter 1: Limits and Their Properties	Limit process
Chapter 2: Differentiation	Calculus

Grace Chisholm Young (1868–1944)

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

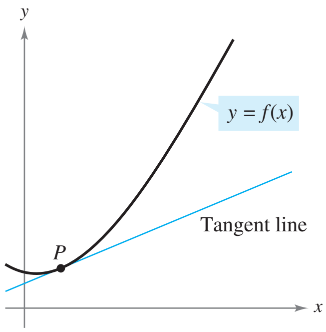
Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	

Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$

The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem and the area problem*—should give you some idea of the way limits are used in calculus.

In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 1.1.



The tangent line to the graph of f at P
Figure 1.1

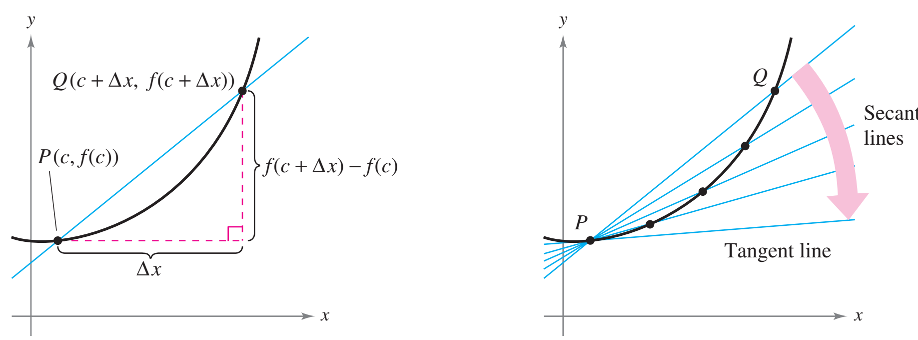
Video

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point P is equivalent to finding the *slope* of the tangent line at P. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, the slope of the secant line through these two points is given by

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

Animation

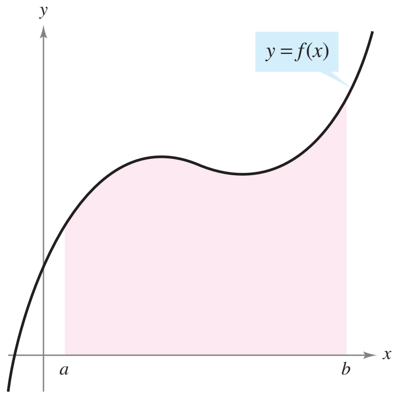
As point Q approaches point P, the slope of the secant line approaches the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slope of the secant line. (Much more will be said about this important problem in Chapter 2.)

EXPLORATION

The following points lie on the graph of $f(x) = x^2$.

$Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$,
$Q_4(1.001, f(1.001))$, $Q_5(1.0001, f(1.0001))$

Each successive point gets closer to the point $P(1, 1)$. Find the slope of the secant line through Q_1 and P, Q_2 and P, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of f at the point P.



Area under a curve
Figure 1.3

Video

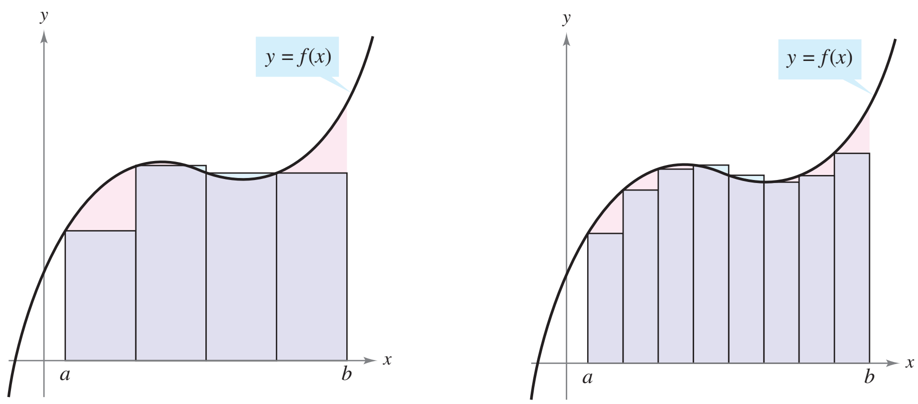
Historical Note

In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.

The Area Problem

In the tangent line problem, you saw how the limit process can be applied to the slope of a line to find the slope of a general curve. A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process. In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1.3. You can approximate the area of the region with several rectangular regions, as shown in Figure 1.4. As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases. Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.

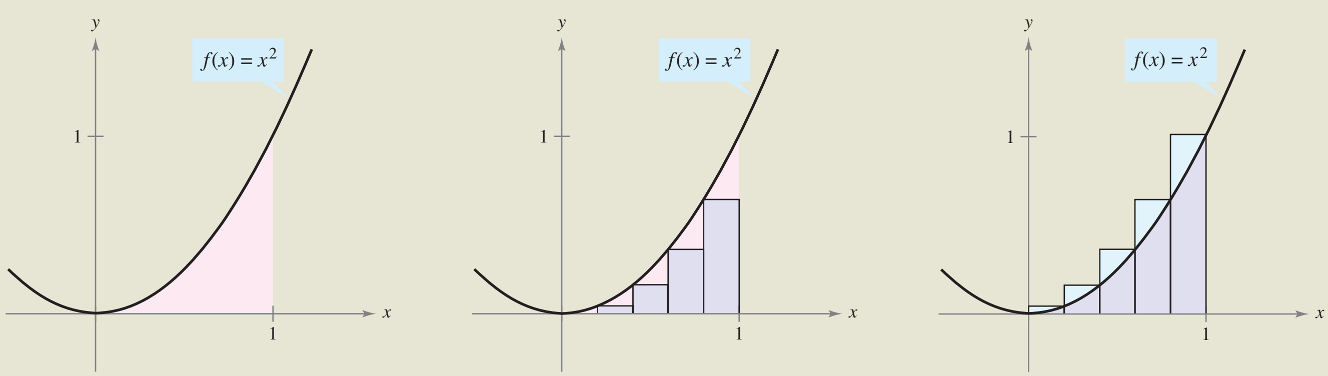


Approximation using four rectangles   Approximation using eight rectangles
Figure 1.4

Animation

EXPLORATION

Consider the region bounded by the graphs of $f(x) = x^2$, $y = 0$, and $x = 1$, as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region   **(b)** Inscribed rectangles   **(c)** Circumscribed rectangles

Exercises for Section 1.1

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on **S** to view the complete solution of the exercise.

Click on **M** to print an enlarged copy of the graph.

In Exercises 1–6, decide whether the problem can be solved using precalculus, or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.

1. Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

2. Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7\cos t$ feet per second.

3. A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation when $x = 2$.

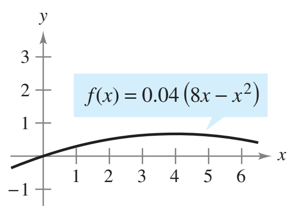


Figure for 3

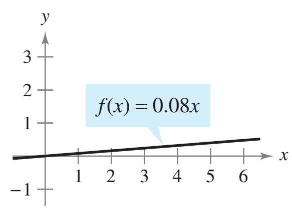


Figure for 4

4. A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation when $x = 2$.

5. Find the area of the shaded region.

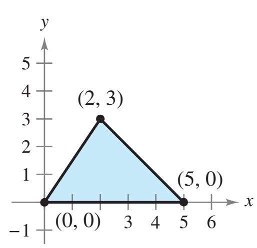


Figure for 5

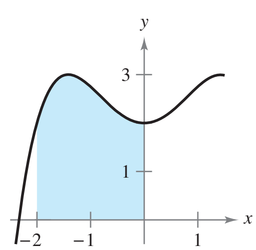


Figure for 6

6. Find the area of the shaded region.

7. ***Secant Lines*** Consider the function $f(x) = 4x - x^2$ and the point $P(1, 3)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(1, 3)$ and $Q(x, f(x))$ for x-values of 2, 1.5, and 0.5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line of f at $P(1, 3)$. Describe how to improve your approximation of the slope.

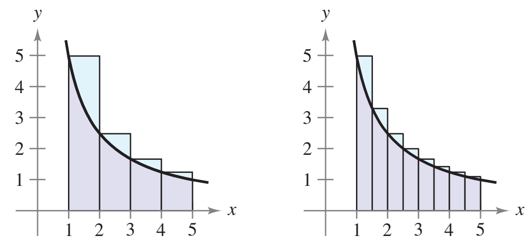
8. ***Secant Lines*** Consider the function $f(x) = \sqrt{x}$ and the point $P(4, 2)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for x-values of 1, 3, and 5.

(b) Find the slope of each secant line.

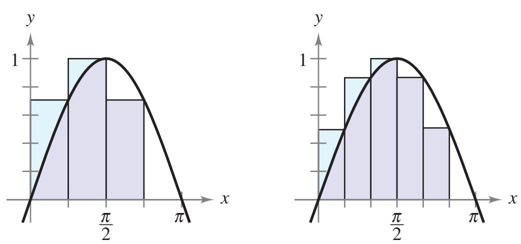
(c) Use the results of part (b) to estimate the slope of th tangent line of f at $P(4, 2)$. Describe how to improve you approximation of the slope.

9. (a) Use the rectangles in each graph to approximate the area o the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$.



(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

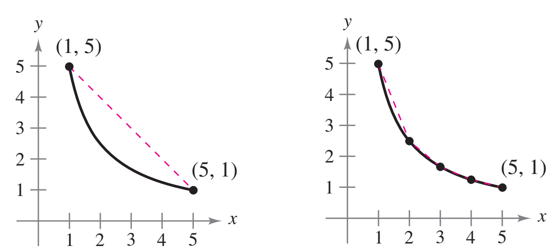
10. (a) Use the rectangles in each graph to approximate the area o the region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$



(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

Writing About Concepts

11. Consider the length of the graph of $f(x) = 5/x$ from $(1, 5)$ to $(5, 1)$.



(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

Section 1.2 Finding Limits Graphically and Numerically

- **Estimate a limit using a numerical or graphical approach.**
- **Learn different ways that a limit can fail to exist.**
- **Study and use a formal definition of limit.**

Video

An Introduction to Limits

Suppose you are asked to sketch the graph of the function f given by

$$f(x) = \frac{x^3 - 1}{x - 1}, \qquad x \neq 1.$$

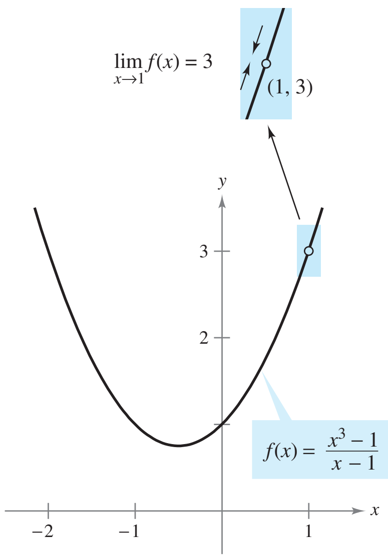
For all values other than $x = 1$, you can use standard curve-sketching techniques. However, at $x = 1$, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.

Animation



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

Animation

The graph of f is a parabola that has a gap at the point $(1, 3)$, as shown in Figure 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x \to 1} f(x) = 3.$$

This is read as "the limit of $f(x)$ as x approaches 1 is 3."

This discussion leads to an informal description of a limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$, as x approaches c, is L. This limit is written as

$$\lim_{x \to c} f(x) = L.$$

EXPLORATION

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/\left(\sqrt{x+1} - 1\right)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

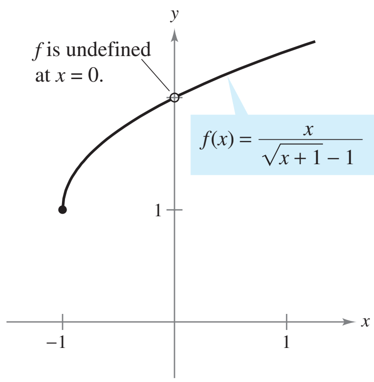
Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. → ← x approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.

From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6).



The limit of $f(x)$ as x approaches 0 is 2.
Figure 1.6

Editable Graph

Try It | Exploration A | Exploration B

In Example 1, note that the function is undefined at $x = 0$ and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2 where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2. \end{cases}$$

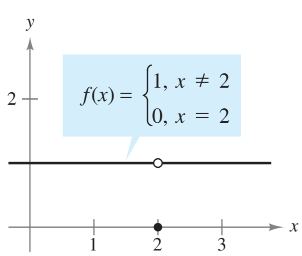
Solution Because $f(x) = 1$ for all x other than $x = 2$, you can conclude that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x \to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, if the function were defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

the limit would be the same.



The limit of $f(x)$ as x approaches 2 is 1.
Figure 1.7

Editable Graph

Try It | Exploration A

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

1. Numerical approach — Construct a table of values.
2. Graphical approach — Draw a graph by hand or using technology.
3. Analytic approach — Use algebra or calculus.

Limits That Fail to Exist

In the next three examples you will examine some limits that fail to exist.

EXAMPLE 3 Behavior That Differs from the Right and Left

Show that the limit does not exist.

$$\lim_{x\to 0} \frac{|x|}{x}$$

Solution Consider the graph of the function $f(x) = |x|/x$. From Figure 1.8, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \qquad x > 0$$

and for negative x-values

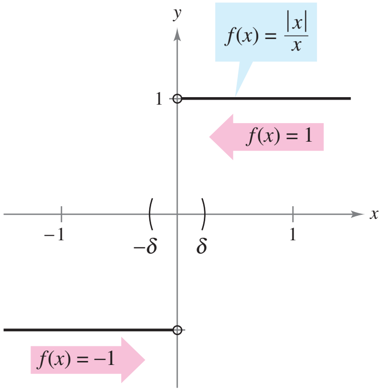
$$\frac{|x|}{x} = -1, \qquad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$. Specifically, if δ (the lowercase Greek letter *delta*) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as shown.

$(-\delta, 0)$ — Negative x-values yield $|x|/x = -1$.

$(0, \delta)$ — Positive x-values yield $|x|/x = 1$.

This implies that the limit does not exist.



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.8

Editable Graph

Try It | Exploration A | Exploration B

EXAMPLE 4 Unbounded Behavior

Discuss the existence of the limit

$$\lim_{x\to 0} \frac{1}{x^2}.$$

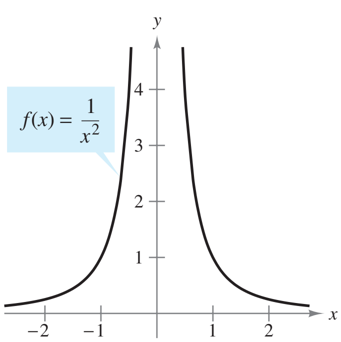
Solution Let $f(x) = 1/x^2$. In Figure 1.9, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist.



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.9

Editable Graph

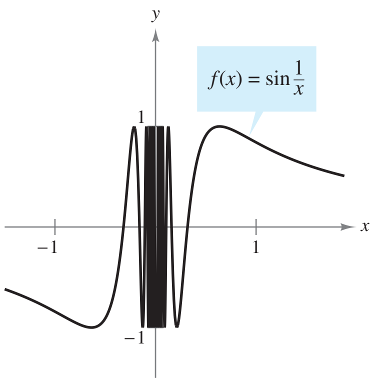
Try It | Exploration A | Exploration B

EXAMPLE 5 Oscillating Behavior

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$2/\pi$	$2/3\pi$	$2/5\pi$	$2/7\pi$	$2/9\pi$	$2/11\pi$	$x \to 0$
sin (1/x)	1	-1	1	-1	1	-1	Limit does not exist.



$\lim_{x \to 0} f(x)$ does not exist.
Figure 1.10

Editable Graph

Try It

Exploration A

Open Exploration

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

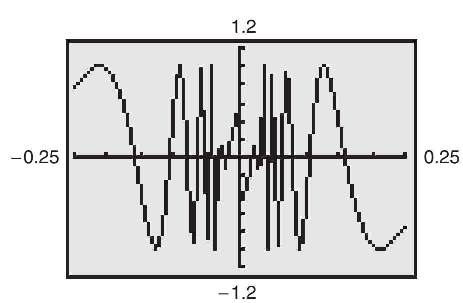
Because this function has *no limit* at any real number c, it is *not continuous* at any real number c. You will study continuity more closely in Section 1.4.

PETER GUSTAV DIRICHLET (1805–1859)

In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of function was given by the German mathematician Peter Gustav Dirichlet.

MathBio

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. If you use a graphing utility to graph the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$.
Figure 1.11

A Formal Definition of Limit

Let's take another look at the informal description of a limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x \to c} f(x) = L.$$

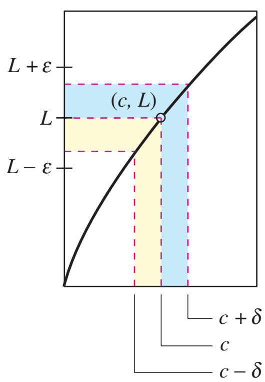
At first glance, this description looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"$f(x)$ becomes arbitrarily close to L"

and

"x approaches c."

The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His **ε-δ definition of limit** is the standard used today.



The ε-δ definition of the limit of $f(x)$ as x approaches c
Figure 1.12

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

The first inequality

$$0 < |x - c| \qquad \text{The distance between } x \text{ and } c \text{ is more than 0.}$$

expresses the fact that $x \neq c$. The second inequality

$$|x - c| < \delta \qquad x \text{ is within } \delta \text{ units of } c.$$

says that x is within a distance δ of c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

NOTE Throughout this text, the expression

$$\lim_{x \to c} f(x) = L$$

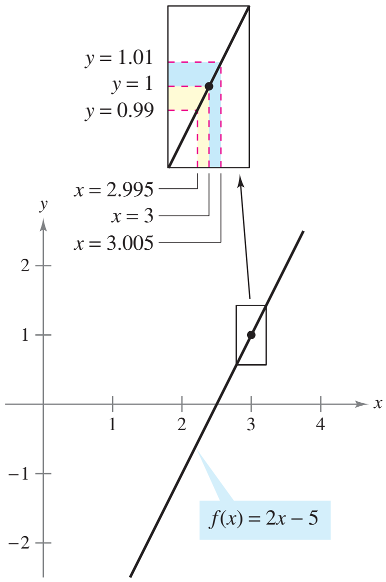
implies two statements—the limit exists *and* the limit is L.

Some functions do not have limits as $x \to c$, but those that do cannot have two different limits as $x \to c$. That is, *if the limit of a function exists, it is unique* (see Exercise 69).

FOR FURTHER INFORMATION For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*.

MathArticle

The next three examples should help you develop a better understanding of the ε-δ definition of limit.



The limit of $f(x)$ as x approaches 3 is 1.
Figure 1.13

EXAMPLE 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$.

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$. This choice works because

$$0 < |x - 3| < 0.005$$

implies that

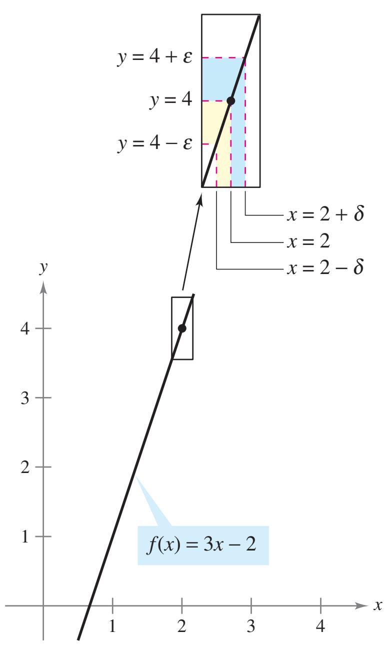
$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01$$

as shown in Figure 1.13.

Try It | Exploration A | Exploration B

NOTE In Example 6, note that 0.005 is the *largest* value of δ that will guarantee $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$. Any *smaller* positive value of δ would also work.

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for any ε, as shown in the next example.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.14

EXAMPLE 7 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} (3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x - 2) - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

So, for a given $\varepsilon > 0$ you can choose $\delta = \varepsilon/3$. This choice works because

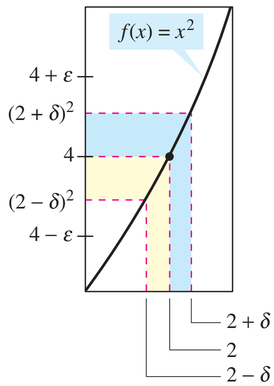
$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

as shown in Figure 1.14.

Try It | Exploration A



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, you know that $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Try It   Exploration A

Throughout this chapter you will use the ε-δ definition of limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of limit.

Exercises for Section 1.2

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\lim_{x\to 2} \dfrac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\lim_{x\to 0} \dfrac{\sqrt{x+3}-\sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4. $\lim_{x\to -3} \dfrac{\sqrt{1-x}-2}{x+3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$						

5. $\lim_{x\to 3} \dfrac{[1/(x+1)]-(1/4)}{x-3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\lim_{x\to 4} \dfrac{[x/(x+1)]-(4/5)}{x-4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\lim_{x\to 0} \dfrac{\sin x}{x}$

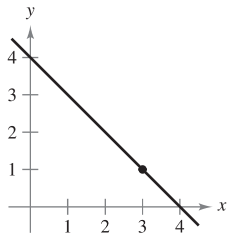
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

8. $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$

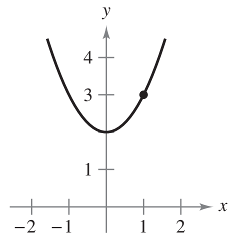
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 9–18, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

9. $\lim_{x\to 3}(4-x)$

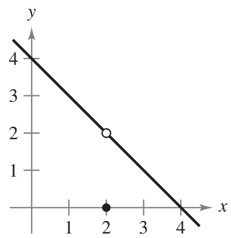


10. $\lim_{x\to 1}(x^2+2)$



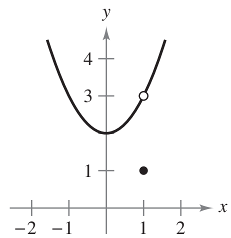
11. $\lim_{x\to 2} f(x)$

$$f(x)=\begin{cases}4-x, & x\neq 2\\ 0, & x=2\end{cases}$$

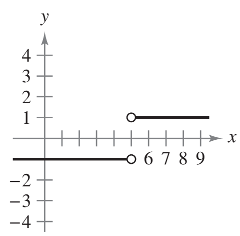


12. $\lim_{x\to 1} f(x)$

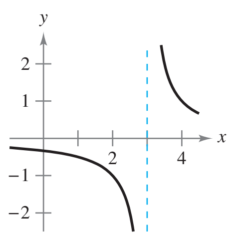
$$f(x)=\begin{cases}x^2+2, & x\neq 1\\ 1, & x=1\end{cases}$$



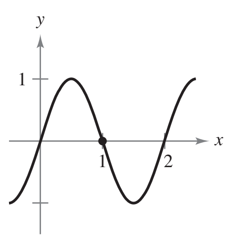
13. $\lim_{x\to 5}\dfrac{|x-5|}{x-5}$



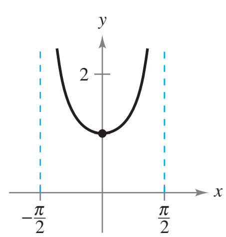
14. $\lim_{x\to 3}\dfrac{1}{x-3}$



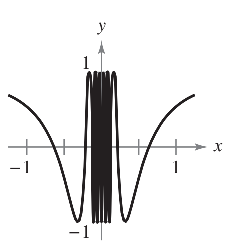
15. $\lim_{x\to 1}\sin \pi x$



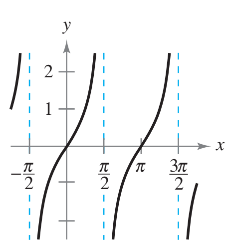
16. $\lim_{x\to 0}\sec x$



17. $\lim_{x\to 0}\cos\dfrac{1}{x}$



18. $\lim_{x\to \pi/2}\tan x$



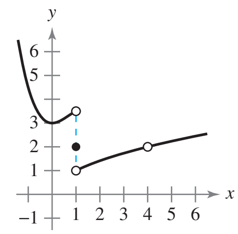
In Exercises 19 and 20, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

19. (a) $f(1)$

(b) $\lim_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to 4} f(x)$



20. (a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(0)$

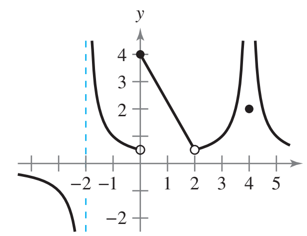
(d) $\lim_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to 2} f(x)$

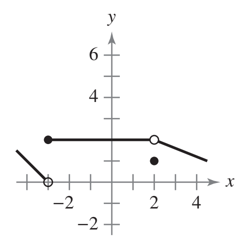
(g) $f(4)$

(h) $\lim_{x\to 4} f(x)$

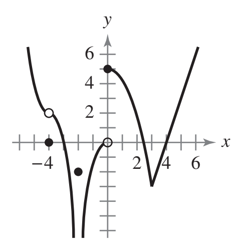


In Exercises 21 and 22, use the graph of f to identify the values of c for which $\lim_{x\to c} f(x)$ exists.

21.



22.



In Exercises 23 and 24, sketch the graph of f. Then identify the values of c for which $\lim_{x\to c} f(x)$ exists.

23. $f(x)=\begin{cases}x^2, & x\le 2\\ 8-2x, & 2<x<4\\ 4, & x\ge 4\end{cases}$

24. $f(x)=\begin{cases}\sin x, & x<0\\ 1-\cos x, & 0\le x\le \pi\\ \cos x, & x>\pi\end{cases}$

In Exercises 25 and 26, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)

25. $f(0)$ is undefined.

$\lim_{x\to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3$

26. $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist.

27. ***Modeling Data*** The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute or fraction thereof. A formula for the cost is given by

$$C(t) = 0.75 - 0.50[\![-(t-1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t\to 3.5} C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

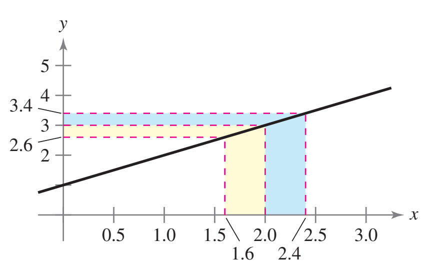
t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

28. Repeat Exercise 27 for

$$C(t) = 0.35 - 0.12[\![-(t-1)]\!].$$

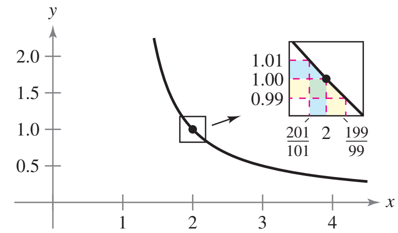
29. The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.4$.



30. The graph of

$$f(x) = \frac{1}{x-1}$$

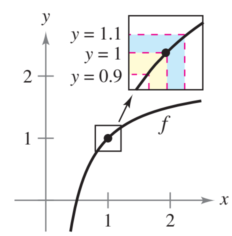
is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 1| < 0.01$.



31. The graph of

$$f(x) = 2 - \frac{1}{x}$$

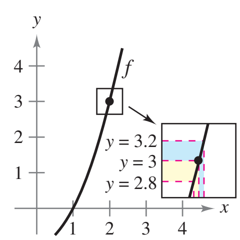
is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$ then $|f(x) - 1| < 0.1$.



32. The graph of

$$f(x) = x^2 - 1$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.2$.



In Exercises 33–36, find the limit L. Then find $\delta > 0$ such tha $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

33. $\lim_{x\to 2} (3x + 2)$

34. $\lim_{x\to 4} \left(4 - \frac{x}{2}\right)$

35. $\lim_{x\to 2} (x^2 - 3)$

36. $\lim_{x\to 5} (x^2 + 4)$

In Exercises 37–48, find the limit L. Then use the ε-δ definition to prove that the limit is L.

37. $\lim_{x\to 2} (x + 3)$

38. $\lim_{x\to -3} (2x + 5)$

39. $\lim_{x\to -4} \left(\frac{1}{2}x - 1\right)$

40. $\lim_{x\to 1} \left(\frac{2}{3}x + 9\right)$

41. $\lim_{x\to 6} 3$

42. $\lim_{x\to 2} (-1)$

43. $\lim_{x\to 0} \sqrt[3]{x}$

44. $\lim_{x\to 4} \sqrt{x}$

45. $\lim_{x\to -2} |x - 2|$

46. $\lim_{x\to 3} |x - 3|$

47. $\lim_{x\to 1} (x^2 + 1)$

48. $\lim_{x\to -3} (x^2 + 3x)$

Writing **In Exercises 49–52, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

49. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

$\lim_{x\to 4} f(x)$

50. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x\to 3} f(x)$

51. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x\to 9} f(x)$

52. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x\to 3} f(x)$

Writing About Concepts

53. Write a brief description of the meaning of the notation $\lim_{x\to 8} f(x) = 25$.

54. If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

55. If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

Writing About Concepts *(continued)*

56. Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

57. ***Jewelry*** A jeweler resizes a ring so that its inner circumferenc is 6 centimeters.

(a) What is the radius of the ring?

(b) If the ring's inner circumference can vary betwee 5.5 centimeters and 6.5 centimeters, how can the radiu vary?

(c) Use the ε-δ definition of limit to describe this situatior Identify ε and δ.

58. ***Sports*** A sporting goods manufacturer designs a golf bal having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) If the ball's volume can vary between 2.45 cubic inches an 2.51 cubic inches, how can the radius vary?

(c) Use the ε-δ definition of limit to describe this situatior Identify ε and δ.

59. Consider the function $f(x) = (1 + x)^{1/x}$. Estimate the limit

$$\lim_{x\to 0} (1 + x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

60. Consider the function

$$f(x) = \frac{|x + 1| - |x - 1|}{x}.$$

Estimate

$$\lim_{x\to 0} \frac{|x + 1| - |x - 1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

61. ***Graphical Analysis*** The statement

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such tha if $0 < |x - 2| < \delta$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Us the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such tha the graph of the left side is below the graph of the right side o the inequality.

62. ***Graphical Analysis*** The statement

$$\lim_{x\to 3} \frac{x^2 - 3x}{x - 3}$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(3 - \delta, 3 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

True or False? **In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

64. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

65. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

66. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

67. Consider the function $f(x) = \sqrt{x}$.

(a) Is $\lim_{x\to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

(b) Is $\lim_{x\to 0} \sqrt{x} = 0$ a true statement? Explain.

68. ***Writing*** The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

69. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit must be unique. [*Hint:* Let

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

70. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x\to c} f(x) = mc + b$.

71. Prove that $\lim_{x\to c} f(x) = L$ is equivalent to $\lim_{x\to c} [f(x) - L] = 0$.

72. (a) Given that

$$\lim_{x\to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing (such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ ii (a, b).

(b) Given that $\lim_{x\to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such tha $g(x) > 0$ for all $x \neq c$ in (a, b).

73. ***Programming*** Use the programming capabilities of a graph ing utility to write a program for approximating $\lim_{x\to c} f(x)$.

Assume the program will be applied only to functions whos limits exist as x approaches c. Let $y_1 = f(x)$ and generate twc lists whose entries form the ordered pairs

$$(c \pm [0.1]^n, f(c \pm [0.1]^n))$$

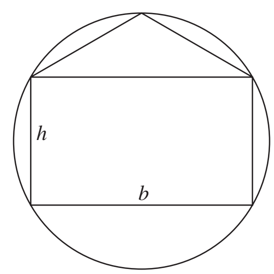
for $n = 0, 1, 2, 3,$ and 4.

74. ***Programming*** Use the program you created in Exercise 73 tc approximate the limit

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}.$$

Putnam Exam Challenge

75. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For wha value of h do the rectangle and triangle have the same area?



76. A right circular cone has base of radius 1 and height 3. A cub is inscribed in the cone so that one face of the cube is containe in the base of the cone. What is the side-length of the cube?

Section 1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using dividing out and rationalizing techniques.
- Evaluate a limit using the Squeeze Theorem.

Video

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c*.** You will examine this concept more closely in Section 1.4.

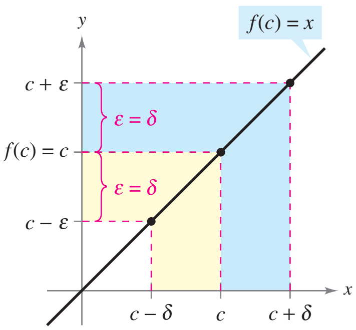


Figure 1.16

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

Proof To prove Property 2 of Theorem 1.1, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. To do this, choose $\delta = \varepsilon$. The second inequality then implies the first, as shown in Figure 1.16. This completes the proof. (Proofs of the other properties of limits in this section are listed in Appendix A or are discussed in the exercises.)

NOTE When you encounter new notations or symbols in mathematics, be sure you know how the notations are read. For instance, the limit in Example 1(c) is read as "the limit of x^2 as x approaches 2 is 4."

EXAMPLE 1 Evaluating Basic Limits

a. $\lim_{x \to 2} 3 = 3$ **b.** $\lim_{x \to -4} x = -4$ **c.** $\lim_{x \to 2} x^2 = 2^2 = 4$

Try It Exploration A

The editable graph feature allows you to edit the graph of a function to visually evaluate the limit as x approaches c.

a. Editable Graph **b.** Editable Graph **c.** Editable Graph

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

EXAMPLE 2 The Limit of a Polynomial

$$\begin{aligned}\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 && \text{Property 2}\\ &= 4\Big(\lim_{x\to 2} x^2\Big) + \lim_{x\to 2} 3 && \text{Property 1}\\ &= 4(2^2) + 3 && \text{Example 1}\\ &= 19 && \text{Simplify.}\end{aligned}$$

Try It | Exploration A

The editable graph feature allows you to edit the graph of a function to visually evaluate the limit as x approaches c.

Editable Graph

In Example 2, note that the limit (as $x \to 2$) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\displaystyle\lim_{x\to 1} \frac{x^2 + x + 2}{x + 1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1} \frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2.$$

Try It | Exploration A

The editable graph feature allows you to edit the graph of a function to visually evaluate the limit as x approaches c.

Editable Graph

Polynomial functions and rational functions are two of the three basic types of algebraic functions. The following theorem deals with the limit of the third type of algebraic function—one that involves a radical. See Appendix A for a proof of this theorem.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase r, to stand for the Latin word *radix*, meaning root.

Video | Video

The following theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function. See Appendix A for a proof of this theorem.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

EXAMPLE 4 The Limit of a Composite Function

a. Because

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4 \qquad \text{and} \qquad \lim_{x \to 4} \sqrt{x} = 2$$

it follows that

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x \to 8} \sqrt[3]{x} = 2$$

it follows that

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

Try It | Exploration A | Open Exploration

The editable graph feature allows you to edit the graph of a function to visually evaluate the limit as x approaches c.

a. Editable Graph **b.** Editable Graph

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$ **2.** $\lim_{x \to c} \cos x = \cos c$

3. $\lim_{x \to c} \tan x = \tan c$ **4.** $\lim_{x \to c} \cot x = \cot c$

5. $\lim_{x \to c} \sec x = \sec c$ **6.** $\lim_{x \to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

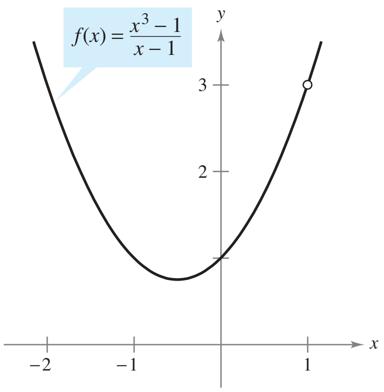
b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

Try It | Exploration A

A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the following theorem, can be used to develop a strategy for finding limits. A proof of this theorem is given in Appendix A.



Editable Graph

THEOREM 1.7 Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

EXAMPLE 6 Finding the Limit of a Function

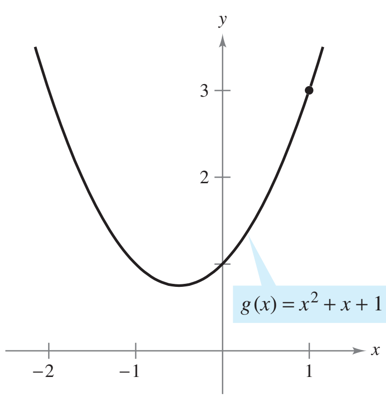
Find the limit: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{\cancel{(x - 1)}(x^2 + x + 1)}{\cancel{(x - 1)}} = x^2 + x + 1 = g(x), \qquad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 1.17. Because $\lim_{x \to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\begin{aligned}
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} && \text{Factor.} \\
&= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x^2 + x + 1)}{\cancel{x - 1}} && \text{Divide out like factors.} \\
&= \lim_{x \to 1} (x^2 + x + 1) && \text{Apply Theorem 1.7.} \\
&= 1^2 + 1 + 1 && \text{Use direct substitution.} \\
&= 3 && \text{Simplify.}
\end{aligned}$$



f and *g* agree at all but one point.

Editable Graph

Figure 1.17

Try It | Exploration A | Exploration B | Exploration C | Exploration D

STUDY TIP When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the following limit does not exist.

$$\lim_{x \to 1} \frac{x^3 + 1}{x - 1}$$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$

4. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out and Rationalizing Techniques

Two techniques for finding limits analytically are shown in Examples 7 and 8. The first technique involves dividing out common factors, and the second technique involves rationalizing the numerator of a fractional expression.

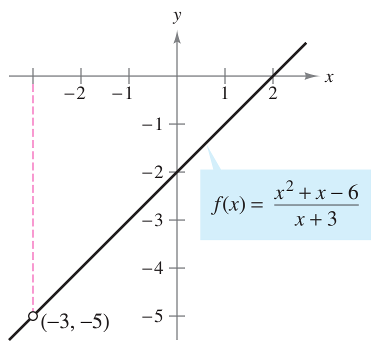
EXAMPLE 7 Dividing Out Technique

Find the limit: $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} \quad \begin{cases} \lim_{x \to -3} (x^2 + x - 6) = 0 \\ \lim_{x \to -3} (x + 3) = 0 \end{cases}$$

Direct substitution fails.



f is undefined when $x = -3$.
Figure 1.18

Editable Graph

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x + 3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{\cancel{(x + 3)}(x - 2)}{\cancel{x + 3}} = x - 2 = g(x), \qquad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} (x - 2) \qquad \text{Apply Theorem 1.7.}$$

$$= -5. \qquad \text{Use direct substitution.}$$

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$.

Try It | Exploration A | Open Exploration

NOTE In the solution of Example 7, be sure you see the usefulness of the Factor Theorem of Algebra. This theorem states that if c is a zero of a polynomial function, $(x - c)$ is a factor of the polynomial. So, if you apply direct substitution to a rational function and obtain

$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0}$$

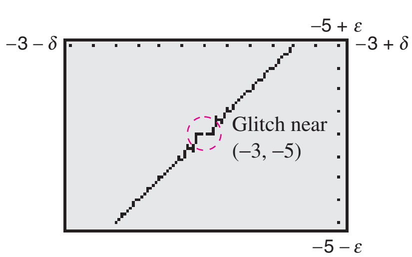
you can conclude that $(x - c)$ must be a common factor to both $p(x)$ and $q(x)$.

In Example 7, direct substitution produced the meaningless fractional form $0/0$. An expression such as $0/0$ is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*, as shown in Example 7. A second way is to *rationalize the numerator*, as shown in Example 8.

TECHNOLOGY PITFALL Because the graphs of

$$f(x) = \frac{x^2 + x - 6}{x + 3} \qquad \text{and} \qquad g(x) = x - 2$$

differ only at the point $(-3, -5)$, a standard graphing utility setting may not distinguish clearly between these graphs. However, because of the pixel configuration and rounding error of a graphing utility, it may be possible to find screen settings that distinguish between the graphs. Specifically, by repeatedly zooming in near the point $(-3, -5)$ on the graph of f, your graphing utility may show glitches or irregularities that do not exist on the actual graph. (See Figure 1.19.) By changing the screen settings on your graphing utility you may obtain the correct graph of f.



Incorrect graph of f
Figure 1.19

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} \quad \begin{cases} \lim_{x \to 0} \left(\sqrt{x+1}-1\right) = 0 \\ \lim_{x \to 0} x = 0 \end{cases}$$

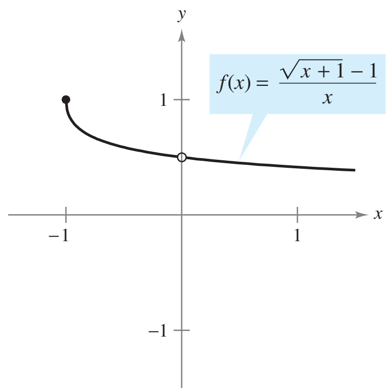
Direct substitution fails.

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\
&= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} \\
&= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} \\
&= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0
\end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

Editable Graph

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.20.)

x approaches 0 from the left. → ← x approaches 0 from the right.

x	−0.25	−0.1	−0.01	−0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

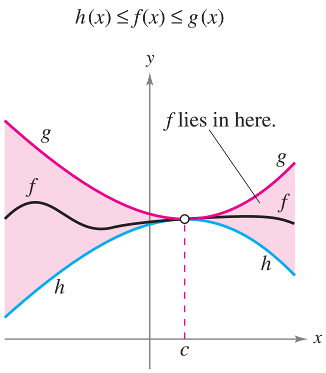
Try It | Exploration A | Exploration B | Exploration C

NOTE The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 8, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21. (The proof of this theorem is given in Appendix A.)



The Squeeze Theorem
Figure 1.21

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

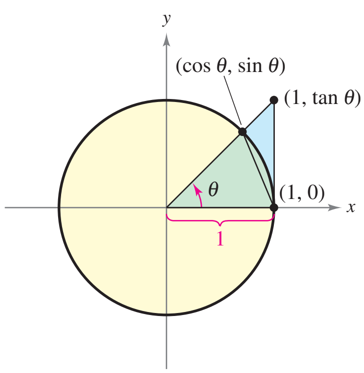
then $\lim_{x\to c} f(x)$ exists and is equal to L.

Video

You can see the usefulness of the Squeeze Theorem in the proof of Theorem 1.9.

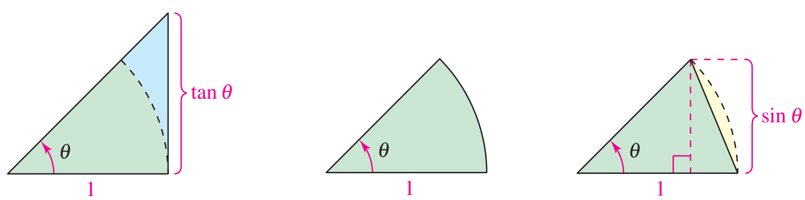
THEOREM 1.9 Two Special Trigonometric Limits

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$



A circular sector is used to prove Theorem 1.9.
Figure 1.22

Proof To avoid the confusion of two different uses of x, the proof is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



Area of triangle	$\ge$	Area of sector	$\ge$	Area of triangle
$\dfrac{\tan\theta}{2}$	$\ge$	$\dfrac{\theta}{2}$	$\ge$	$\dfrac{\sin\theta}{2}$

Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta\to 0} \cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$. The proof of the second limit is left as an exercise (see Exercise 120).

FOR FURTHER INFORMATION For more information on the function $f(x) = (\sin x)/x$, see the article "The Function $(\sin x)/x$" by William B. Gearhart and Harris S. Shultz in *The College Mathematics Journal*.

MathArticle

EXAMPLE 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

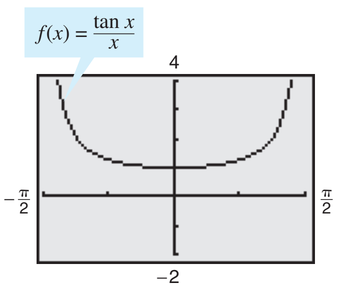
Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

Editable Graph

Try It

Exploration A

EXAMPLE 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

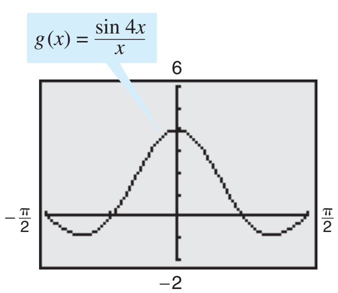
Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right). \qquad \text{Multiply and divide by 4.}$$

Now, by letting $y = 4x$ and observing that $x \to 0$ if and only if $y \to 0$, you can write

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) \\ &= 4(1) \\ &= 4. \end{aligned}$$

(See Figure 1.24.)



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

Editable Graph

Try It

Exploration A

TECHNOLOGY Use a graphing utility to confirm the limits in the examples and exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \qquad \text{and} \qquad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second graph appears to contain the point (0, 4), which lends support to the conclusions obtained in Examples 9 and 10.

Exercises for Section 1.3

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = x^2 - 5x$
   (a) $\lim_{x\to 5} h(x)$
   (b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
   (a) $\lim_{x\to 4} g(x)$
   (b) $\lim_{x\to 0} g(x)$

3. $f(x) = x \cos x$
   (a) $\lim_{x\to 0} f(x)$
   (b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
   (a) $\lim_{t\to 4} f(t)$
   (b) $\lim_{t\to -1} f(t)$

In Exercises 5–22, find the limit.

5. $\lim_{x\to 2} x^4$
6. $\lim_{x\to -2} x^3$
7. $\lim_{x\to 0} (2x - 1)$
8. $\lim_{x\to -3} (3x + 2)$
9. $\lim_{x\to -3} (x^2 + 3x)$
10. $\lim_{x\to 1} (-x^2 + 1)$
11. $\lim_{x\to -3} (2x^2 + 4x + 1)$
12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$
13. $\lim_{x\to 2} \dfrac{1}{x}$
14. $\lim_{x\to -3} \dfrac{2}{x + 2}$
15. $\lim_{x\to 1} \dfrac{x - 3}{x^2 + 4}$
16. $\lim_{x\to 3} \dfrac{2x - 3}{x + 5}$
17. $\lim_{x\to 7} \dfrac{5x}{\sqrt{x + 2}}$
18. $\lim_{x\to 3} \dfrac{\sqrt{x + 1}}{x - 4}$
19. $\lim_{x\to 3} \sqrt{x + 1}$
20. $\lim_{x\to 4} \sqrt[3]{x + 4}$
21. $\lim_{x\to -4} (x + 3)^2$
22. $\lim_{x\to 0} (2x - 1)^3$

In Exercises 23–26, find the limits.

23. $f(x) = 5 - x,\ g(x) = x^3$
    (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
    (a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
    (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
    (a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x\to \pi/2} \sin x$
28. $\lim_{x\to \pi} \tan x$
29. $\lim_{x\to 2} \cos \dfrac{\pi x}{3}$
30. $\lim_{x\to 1} \sin \dfrac{\pi x}{2}$
31. $\lim_{x\to 0} \sec 2x$
32. $\lim_{x\to \pi} \cos 3x$
33. $\lim_{x\to 5\pi/6} \sin x$
34. $\lim_{x\to 5\pi/3} \cos x$
35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x\to c} f(x) = 2$
    $\lim_{x\to c} g(x) = 3$
    (a) $\lim_{x\to c} [5g(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

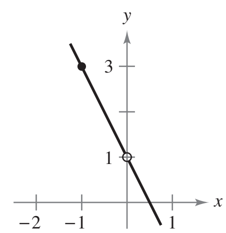
38. $\lim_{x\to c} f(x) = \frac{3}{2}$
    $\lim_{x\to c} g(x) = \frac{1}{2}$
    (a) $\lim_{x\to c} [4f(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
    (a) $\lim_{x\to c} [f(x)]^3$
    (b) $\lim_{x\to c} \sqrt{f(x)}$
    (c) $\lim_{x\to c} [3f(x)]$
    (d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
    (a) $\lim_{x\to c} \sqrt[3]{f(x)}$
    (b) $\lim_{x\to c} \dfrac{f(x)}{18}$
    (c) $\lim_{x\to c} [f(x)]^2$
    (d) $\lim_{x\to c} [f(x)]^{2/3}$

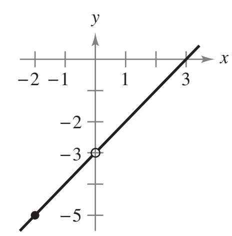
In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

41. $g(x) = \dfrac{-2x^2 + x}{x}$



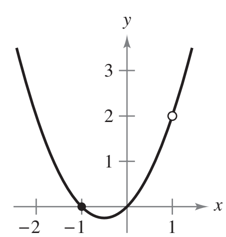
   (a) $\lim_{x\to 0} g(x)$
   (b) $\lim_{x\to -1} g(x)$

42. $h(x) = \dfrac{x^2 - 3x}{x}$



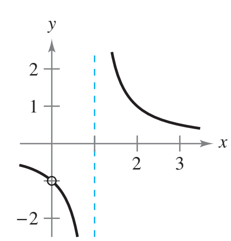
   (a) $\lim_{x\to -2} h(x)$
   (b) $\lim_{x\to 0} h(x)$

43. $g(x) = \dfrac{x^3 - x}{x - 1}$



   (a) $\lim_{x\to 1} g(x)$
   (b) $\lim_{x\to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



   (a) $\lim_{x\to 1} f(x)$
   (b) $\lim_{x\to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

46. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1}$

47. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

48. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

In Exercises 49–62, find the limit (if it exists).

49. $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$

50. $\lim_{x \to 2} \frac{2 - x}{x^2 - 4}$

51. $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9}$

52. $\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8}$

53. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

54. $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$

55. $\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

56. $\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3}$

57. $\lim_{x \to 0} \frac{[1/(3 + x)] - (1/3)}{x}$

58. $\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$

59. $\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x}$

60. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$

61. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}$

62. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$

Graphical, Numerical, and Analytic Analysis **In Exercises 63–66, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

63. $\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x}$

64. $\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16}$

65. $\lim_{x \to 0} \frac{[1/(2 + x)] - (1/2)}{x}$

66. $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

In Exercises 67–78, determine the limit of the trigonometric function (if it exists).

67. $\lim_{x \to 0} \frac{\sin x}{5x}$

68. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x}$

69. $\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{2x^2}$

70. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta}$

71. $\lim_{x \to 0} \frac{\sin^2 x}{x}$

72. $\lim_{x \to 0} \frac{\tan^2 x}{x}$

73. $\lim_{h \to 0} \frac{(1 - \cos h)^2}{h}$

74. $\lim_{\phi \to \pi} \phi \sec \phi$

75. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$

76. $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

77. $\lim_{t \to 0} \frac{\sin 3t}{2t}$

78. $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint}\text{: Find } \lim_{x \to 0} \left(\frac{2 \sin 2x}{2x}\right)\left(\frac{3x}{3 \sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 79–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

79. $\lim_{t \to 0} \frac{\sin 3t}{t}$

80. $\lim_{x \to 0} \frac{\cos x - 1}{2x^2}$

81. $\lim_{x \to 0} \frac{\sin x^2}{x}$

82. $\lim_{x \to 0} \frac{\sin x}{\sqrt[3]{x}}$

In Exercises 83–86, find $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

83. $f(x) = 2x + 3$

84. $f(x) = \sqrt{x}$

85. $f(x) = \frac{4}{x}$

86. $f(x) = x^2 - 4x$

In Exercises 87 and 88, use the Squeeze Theorem to find $\lim_{x \to c} f(x)$.

87. $c = 0$

$4 - x^2 \le f(x) \le 4 + x^2$

88. $c = a$

$b - |x - a| \le f(x) \le b + |x - a|$

In Exercises 89–94, use a graphing utility to graph the given function and the equations $y = |x|$ and $y = -|x|$ in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x \to 0} f(x)$.

89. $f(x) = x \cos x$

90. $f(x) = |x \sin x|$

91. $f(x) = |x| \sin x$

92. $f(x) = |x| \cos x$

93. $f(x) = x \sin \frac{1}{x}$

94. $h(x) = x \cos \frac{1}{x}$

Writing About Concepts

95. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

96. Give an example of two functions that agree at all but one point.

97. What is meant by an indeterminate form?

98. In your own words, explain the Squeeze Theorem.

99. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x \to 0} h(x) = 1.$$

100. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \ g(x) = \sin^2 x, \text{ and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x \to 0} h(x) = 0.$$

Free-Falling Object **In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 1000$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 1000 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

101. If a construction worker drops a wrench from a height of 1000 feet, how fast will the wrench be falling after 5 seconds?

102. If a construction worker drops a wrench from a height of 1000 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 150$, which gives the height (in meters) of an object that has fallen from a height of 150 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but $\lim_{x \to 0} [f(x) + g(x)]$ does exist.

106. Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

107. Prove Property 1 of Theorem 1.1.

108. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Prove Property 1 of Theorem 1.2.

110. Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

111. Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x \to c} f(x)g(x) = 0$.

112. (a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.
(*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$.
[*Hint:* Use the inequality $\big||f(x)| - |L|\big| \le |f(x) - L|$.]

True or False? **In Exercises 113–118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

113. $\lim_{x \to 0} \frac{|x|}{x} = 1$

114. $\lim_{x \to \pi} \frac{\sin x}{x} = 1$

115. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and

$\lim_{x \to 0} f(x) = L$, then $\lim_{x \to 0} g(x) = L$.

116. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

117. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

118. If $f(x) < g(x)$ for all $x \ne a$, then

$\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$.

119. ***Think About It*** Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

120. Prove the second part of Theorem 1.9 by proving that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

121. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

122. ***Graphical Reasoning*** Consider $f(x) = \frac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm the answer in part (c) analytically.

123. ***Approximation***

(a) Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

(b) Use the result in part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use the result in part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

124. ***Think About It*** When using a graphing utility to generate a table to approximate $\lim_{x \to 0} [(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

Section 1.4 Continuity and One-Sided Limits

- **Determine continuity at a point and continuity on an open interval.**
- **Determine one-sided limits and continuity on a closed interval.**
- **Use properties of continuity.**
- **Understand and use the Intermediate Value Theorem.**

EXPLORATION

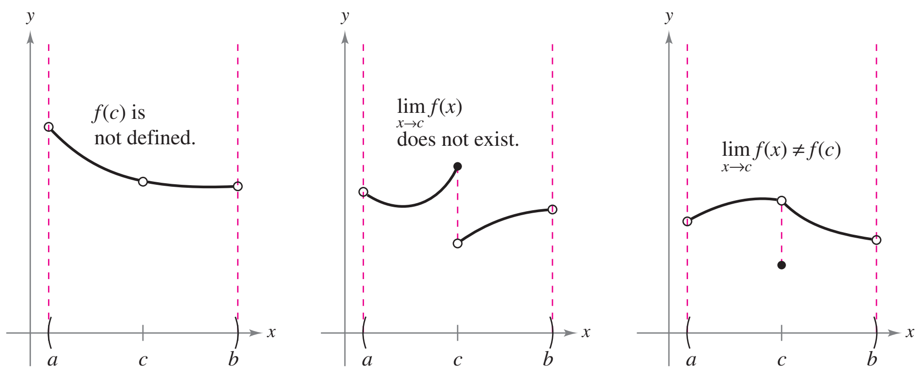
Informally, you might say that a function is *continuous* on an open interval if its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing utility to graph each function on the given interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

Function	*Interval*
a. $y = x^2 + 1$	$(-3, 3)$
b. $y = \dfrac{1}{x - 2}$	$(-3, 3)$
c. $y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d. $y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$
e. $y = \begin{cases} 2x - 4, & x \le 0 \\ x + 1, & x > 0 \end{cases}$	$(-3, 3)$

Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are no holes, jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is *not* continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and **continuous.**

Animation



Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the above three conditions is true, the function f is called **continuous at c,** as indicated in the following important definition.

FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*.

MathArticle

Definition of Continuity

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

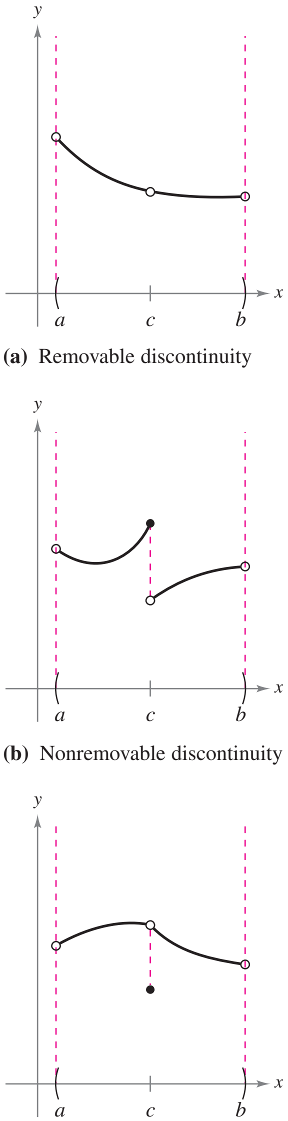
1. $f(c)$ is defined.

2. $\lim_{x \to c} f(x)$ exists.

3. $\lim_{x \to c} f(x) = f(c)$.

Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**

Video



(a) Removable discontinuity

(b) Nonremovable discontinuity

(c) Removable discontinuity

Figure 1.26

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figure 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.

EXAMPLE 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

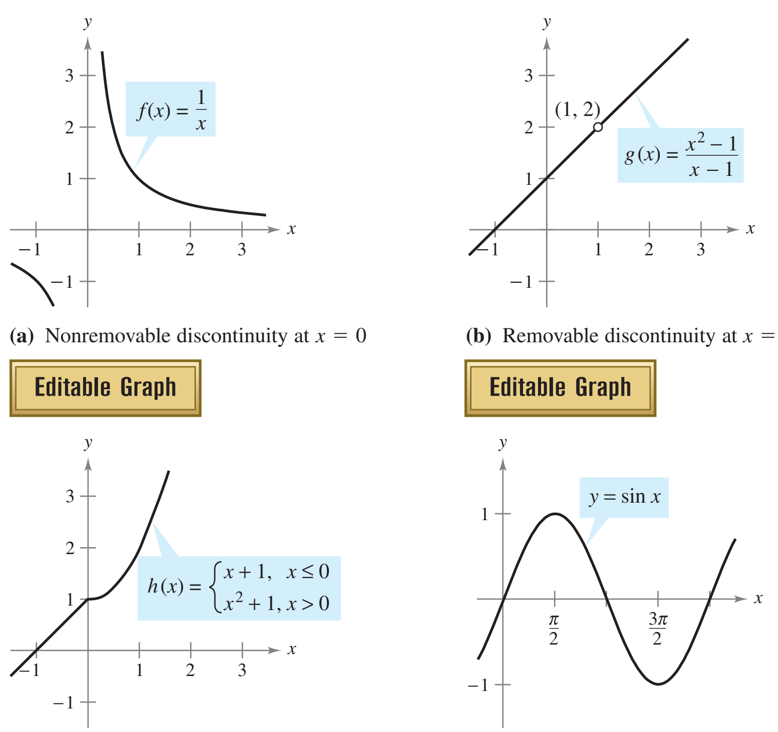
Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because $\lim_{x \to 0} h(x) = 1$, h is continuous on the entire real line, as shown in Figure 1.27(c).

d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



(a) Nonremovable discontinuity at $x = 0$

Editable Graph

(b) Removable discontinuity at $x = 1$

Editable Graph

(c) Continuous on entire real line

Editable Graph

(d) Continuous on entire real line

Editable Graph

Figure 1.27

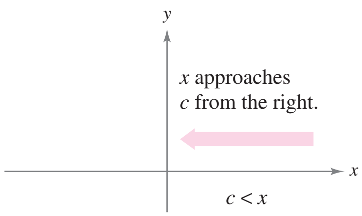
STUDY TIP Some people may refer to the function in Example 1(a) as "discontinuous." We have found that this terminology can be confusing. Rather than saying the function is discontinuous, we prefer to say that it has a discontinuity at $x = 0$.

Try It | Exploration A | Exploration B | Exploration C

One-Sided Limits and Continuity on a Closed Interval

To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.** For example, the **limit from the right** means that x approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted as

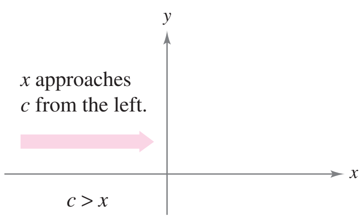
$$\lim_{x\to c^+} f(x) = L. \qquad \text{Limit from the right}$$



(a) Limit from right

Similarly, the **limit from the left** means that x approaches c from values less than c [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x\to c^-} f(x) = L. \qquad \text{Limit from the left}$$



(b) Limit from left
Figure 1.28

One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer,

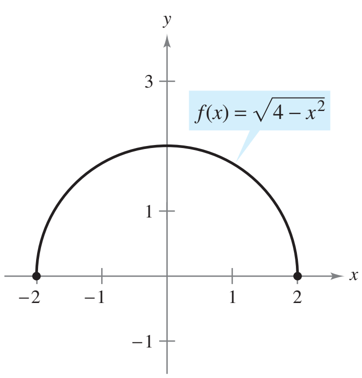
$$\lim_{x\to 0^+} \sqrt[n]{x} = 0.$$

EXAMPLE 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution As shown in Figure 1.29, the limit as x approaches -2 from the right is

$$\lim_{x\to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.
Figure 1.29

Editable Graph

Try It Exploration A

One-sided limits can be used to investigate the behavior of **step functions.** One common type of step function is the **greatest integer function** $[\![x]\!]$, defined by

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x. \qquad \text{Greatest integer function}$$

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.

EXAMPLE 3 The Greatest Integer Function

Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from the left and from the right.

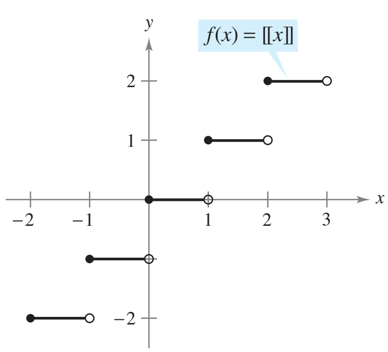
Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is given by

$$\lim_{x\to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is given by

$$\lim_{x\to 0^+} [\![x]\!] = 0.$$

The greatest integer function has a discontinuity at zero because the left and right limits at zero are different. By similar reasoning, you can see that the greatest integer function has a discontinuity at any integer n.



Greatest integer function
Figure 1.30

Editable Graph

Try It Exploration A Exploration B

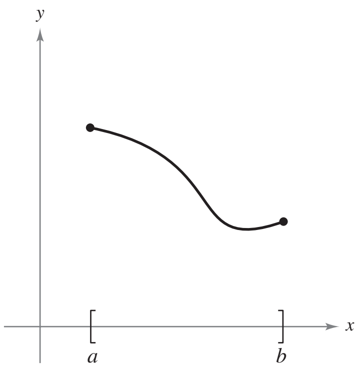
When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval if it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally as follows.



Continuous function on a closed interval
Figure 1.31

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

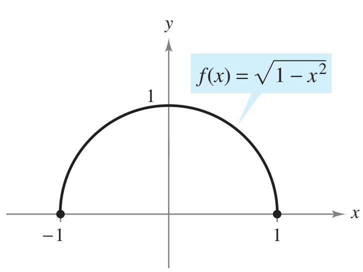
is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.



f is continuous on $[-1, 1]$.
Figure 1.32

Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32.

Editable Graph

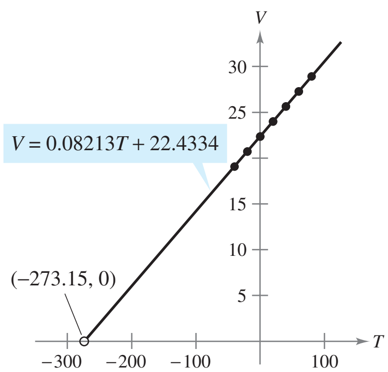
Try It

Exploration A

The next example shows how a one-sided limit can be used to determine the value of absolute zero on the Kelvin scale.

EXAMPLE 5 Charles's Law and Absolute Zero

On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperatures of approximately 0.0001 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero *cannot* be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?



The volume of hydrogen gas depends on its temperature.
Figure 1.33

Editable Graph

Solution The determination of absolute zero stems from the work of the French physicist Jacques Charles (1746–1823). Charles discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. The table illustrates this relationship between volume and temperature. In the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is measured in liters and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

The points represented by the table are shown in Figure 1.33. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

$$V = 0.08213T + 22.4334 \qquad \text{or} \qquad T = \frac{V - 22.4334}{0.08213}.$$

By reasoning that the volume of the gas can approach 0 (but never equal or go below 0) you can determine that the "least possible temperature" is given by

$$\begin{aligned}\lim_{V\to 0^+} T &= \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213}\\ &= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.}\\ &\approx -273.15.\end{aligned}$$

So, absolute zero on the Kelvin scale (0 K) is approximately $-273.15°$ on the Celsius scale.

Try It   Exploration A

The following table shows the temperatures in Example 5, converted to the Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperatures and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale.

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

In 1995, physicists Carl Wieman and Eric Cornell of the University of Colorado at Boulder used lasers and evaporation to produce a supercold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. "We get to within a billionth of a degree of absolute zero," reported Wieman. (*Source:* Time *magazine, April 10, 2000*)

NOTE Charles's Law for gases (assuming constant pressure) can be stated as

$$V = RT \qquad \text{Charles's Law}$$

where V is volume, R is constant, and T is temperature. In the statement of this law, what property must the temperature scale have?

AUGUSTIN-LOUIS CAUCHY (1789–1857)

The concept of a continuous function was first introduced by Augustin-Louis Cauchy in 1821. The definition given in his text *Cours d'Analyse* stated that indefinite small changes in y were the result of indefinite small changes in x. "… $f(x)$ will be called a *continuous* function if … the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α …."

MathBio

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2.

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$
2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$
4. Trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

EXAMPLE 6 Applying Properties of Continuity

By Theorem 1.11, it follows that each of the following functions is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3\tan x, \quad f(x) = \frac{x^2 + 1}{\cos x}$$

Try It | Exploration A | Open Exploration

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \quad f(x) = \sqrt{x^2 + 1}, \quad f(x) = \tan\frac{1}{x}.$$

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

One consequence of Theorem 1.12 is that if f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

Technology

EXAMPLE 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \qquad n \text{ is an integer.}$$

At all other points it is continuous. So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Section 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).

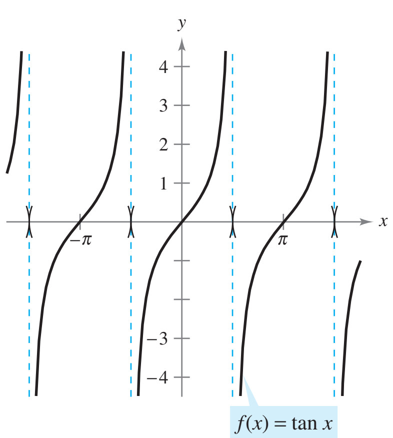
c. This function is similar to that in part (b) except that the oscillations are damped by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \qquad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

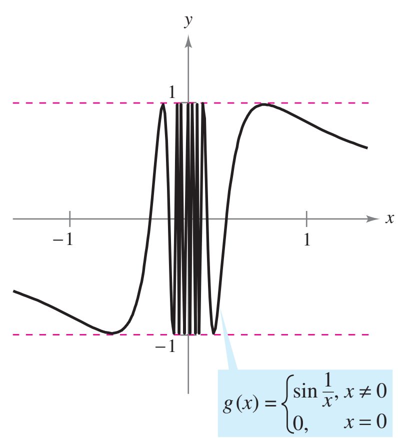
So, h is continuous on the entire real line, as shown in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

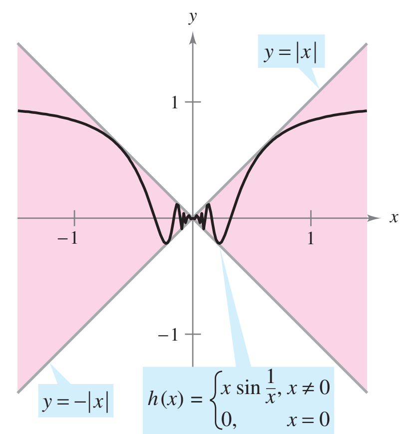
Editable Graph

Figure 1.34



(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

Editable Graph



(c) h is continuous on the entire real line.

Editable Graph

Try It

Exploration A

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

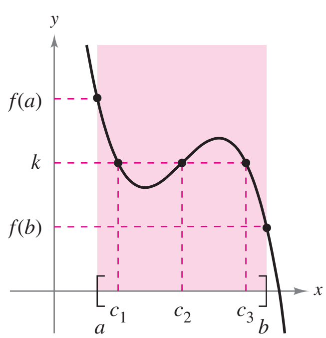
Video

NOTE The Intermediate Value Theorem tells you that at least one c exists, but it does not give a method for finding c. Such theorems are called **existence theorems.**

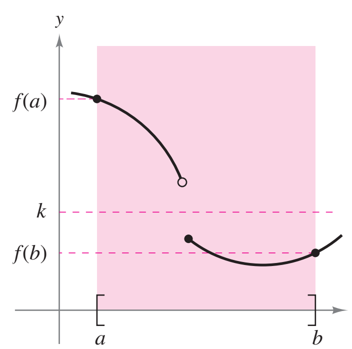
By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 1.35. A function that is not continuous does not necessarily exhibit the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.

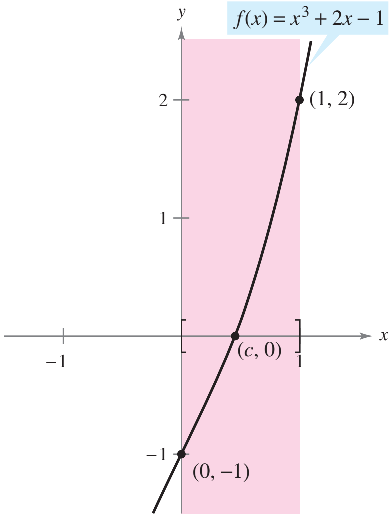


f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]
Figure 1.35



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]
Figure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.
Figure 1.37

Editable Graph

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37.

Try It Exploration A

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

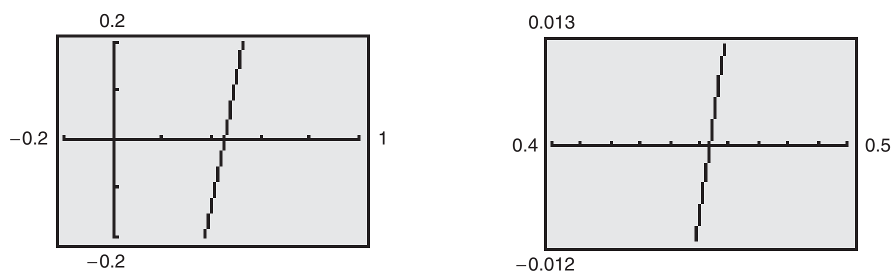


Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

Exercises for Section 1.4

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

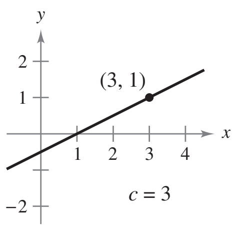
Click on **S** to view the complete solution of the exercise.

Click on **M** to print an enlarged copy of the graph.

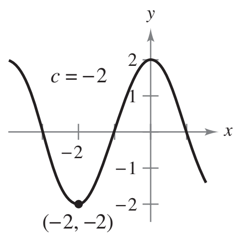
In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ **(b)** $\lim_{x \to c^-} f(x)$ **(c)** $\lim_{x \to c} f(x)$

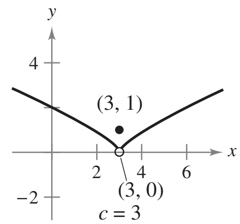
1.



2.



3.



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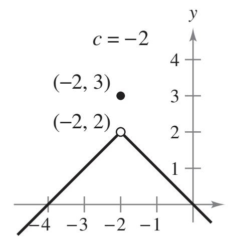

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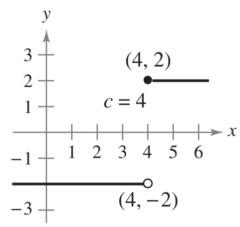

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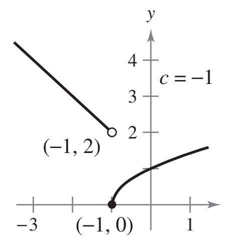


In Exercises 7–24, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25}$

8. $\lim_{x\to 2^+} \dfrac{2-x}{x^2-4}$

9. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

10. $\lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$

11. $\lim_{x\to 0^-} \dfrac{|x|}{x}$

12. $\lim_{x\to 2^+} \dfrac{|x-2|}{x-2}$

13. $\lim_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}$

14. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+x+\Delta x-(x^2+x)}{\Delta x}$

15. $\lim_{x\to 3^-} f(x)$, where $f(x)=\begin{cases}\dfrac{x+2}{2}, & x\le 3\\[4pt] \dfrac{12-2x}{3}, & x>3\end{cases}$

16. $\lim_{x\to 2} f(x)$, where $f(x)=\begin{cases}x^2-4x+6, & x<2\\ -x^2+4x-2, & x\ge 2\end{cases}$

17. $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^3+1, & x<1\\ x+1, & x\ge 1\end{cases}$

18. $\lim_{x\to 1^+} f(x)$, where $f(x)=\begin{cases}x, & x\le 1\\ 1-x, & x>1\end{cases}$

19. $\lim_{x\to \pi} \cot x$

20. $\lim_{x\to \pi/2} \sec x$

21. $\lim_{x\to 4^-} (3[\![x]\!]-5)$

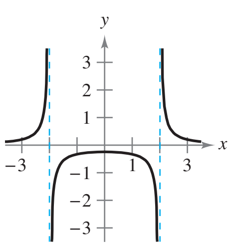
22. $\lim_{x\to 2^+} (2x-[\![x]\!])$

23. $\lim_{x\to 3} (2-[\![-x]\!])$

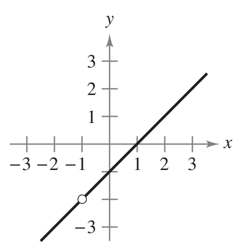
24. $\lim_{x\to 1} \left(1-\left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

In Exercises 25–28, discuss the continuity of each function.

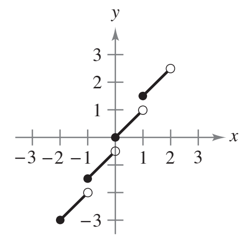
25. $f(x)=\dfrac{1}{x^2-4}$



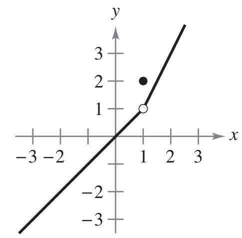
26. $f(x)=\dfrac{x^2-1}{x+1}$



27. $f(x)=\frac{1}{2}[\![x]\!]+x$



28. $f(x)=\begin{cases}x, & x<1\\ 2, & x=1\\ 2x-1, & x>1\end{cases}$



In Exercises 29–32, discuss the continuity of the function on the closed interval.

Function	*Interval*
29. $g(x)=\sqrt{25-x^2}$	$[-5, 5]$
30. $f(t)=3-\sqrt{9-t^2}$	$[-3, 3]$
31. $f(x)=\begin{cases}3-x, & x\le 0\\ 3+\frac{1}{2}x, & x>0\end{cases}$	$[-1, 4]$
32. $g(x)=\dfrac{1}{x^2-4}$	$[-1, 2]$

In Exercises 33–54, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

33. $f(x)=x^2-2x+1$

34. $f(x)=\dfrac{1}{x^2+1}$

35. $f(x)=3x-\cos x$

36. $f(x)=\cos\dfrac{\pi x}{2}$

37. $f(x)=\dfrac{x}{x^2-x}$

38. $f(x)=\dfrac{x}{x^2-1}$

39. $f(x)=\dfrac{x}{x^2+1}$

40. $f(x)=\dfrac{x-3}{x^2-9}$

41. $f(x)=\dfrac{x+2}{x^2-3x-10}$

42. $f(x)=\dfrac{x-1}{x^2+x-2}$

43. $f(x)=\dfrac{|x+2|}{x+2}$

44. $f(x)=\dfrac{|x-3|}{x-3}$

45. $f(x)=\begin{cases}x, & x\le 1\\ x^2, & x>1\end{cases}$

46. $f(x)=\begin{cases}-2x+3, & x<1\\ x^2, & x\ge 1\end{cases}$

47. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

50. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \frac{\pi x}{2}$

53. $f(x) = [\![x - 1]\!]$

54. $f(x) = 3 - [\![x]\!]$

In Exercises 55 and 56, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

55. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

56. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 57–60, find the constant *a*, or the constants *a* and *b*, such that the function is continuous on the entire real line.

57. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

58. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

59. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

60. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 61–64, discuss the continuity of the composite function $h(x) = f(g(x))$.

61. $f(x) = x^2$
$g(x) = x - 1$

62. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

63. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

64. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 65–68, use a graphing utility to graph the function Use the graph to determine any *x*-values at which the functio is not continuous.

65. $f(x) = [\![x]\!] - x$

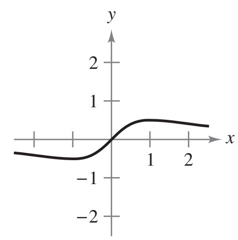
66. $h(x) = \dfrac{1}{x^2 - x - 2}$

67. $g(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

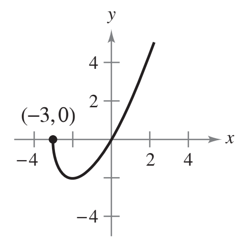
68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 69–72, describe the interval(s) on which th function is continuous.

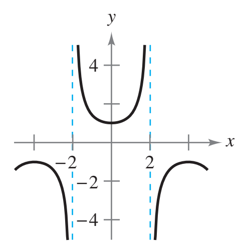
69. $f(x) = \dfrac{x}{x^2 + 1}$



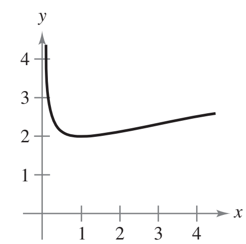
70. $f(x) = x\sqrt{x + 3}$



71. $f(x) = \sec \dfrac{\pi x}{4}$



72. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 73 and 74, use a graphing utility to grapl the function on the interval $[-4, 4]$. Does the graph of the func tion appear continuous on this interval? Is the function contin uous on $[-4, 4]$? Write a short paragraph about the importanc of examining a function analytically as well as graphically.**

73. $f(x) = \dfrac{\sin x}{x}$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 75–78, explain why the function has zero in the given interval.**

	Function	*Interval*
75.	$f(x) = \frac{1}{16}x^4 - x^3 + 3$	$[1, 2]$
76.	$f(x) = x^3 + 3x - 2$	$[0, 1]$
77.	$f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
78.	$f(x) = -\frac{4}{x} + \tan\left(\frac{\pi x}{8}\right)$	$[1, 3]$

In Exercises 79–82, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

79. $f(x) = x^3 + x - 1$

80. $f(x) = x^3 + 3x - 2$

81. $g(t) = 2\cos t - 3t$

82. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 83–86, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of *c* guaranteed by the theorem.

83. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

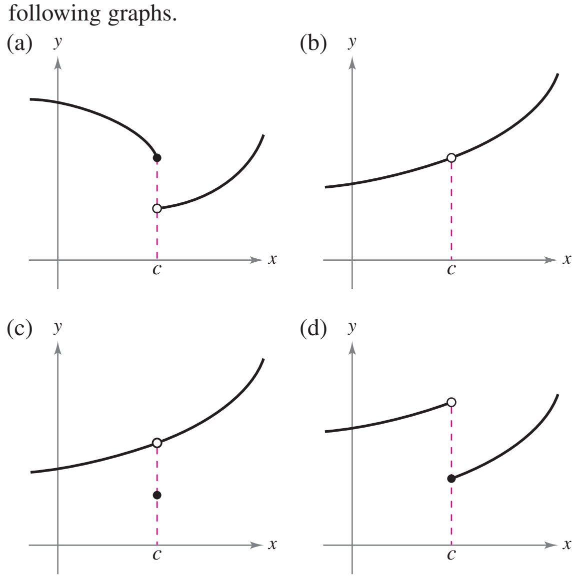
84. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

85. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

86. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

Writing About Concepts

87. State how continuity is destroyed at $x = c$ for each of the following graphs.



88. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 2$

(b) A function with a removable discontinuity at $x = -2$

(c) A function that has both of the characteristics described in parts (a) and (b)

Writing About Concepts *(continued)*

89. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

90. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

True or False? **In Exercises 91–94, determine whether th statement is true or false. If it is false, explain why or give a example that shows it is false.**

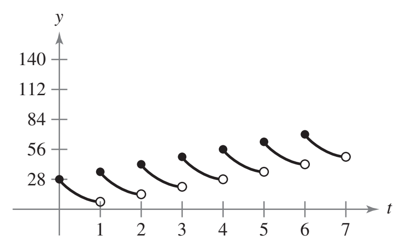
91. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

92. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g i not continuous at c.

93. A rational function can have infinitely many x-values at whic it is not continuous.

94. The function $f(x) = |x - 1|/(x - 1)$ is continuous o $(-\infty, \infty)$.

95. *Swimming Pool* Every day you dissolve 28 ounces o chlorine in a swimming pool. The graph shows the amount o chlorine $f(t)$ in the pool after t days.



Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.

96. *Think About It* Describe how the functions

$f(x) = 3 + [\![x]\!]$

and

$g(x) = 3 - [\![-x]\!]$

differ.

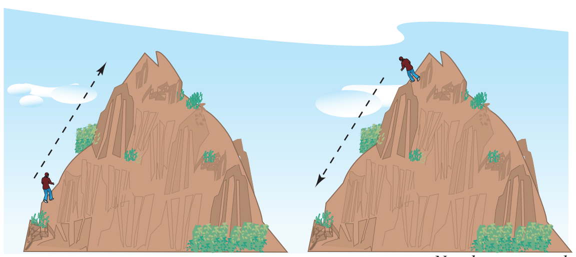
97. *Telephone Charges* A dial-direct long distance call betwee two cities costs \$1.04 for the first 2 minutes and \$0.36 for eac additional minute or fraction thereof. Use the greatest intege function to write the cost C of a call in terms of time t (i minutes). Sketch the graph of this function and discuss it continuity.

98. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

99. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

100. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[1, 5]$, there is one with a volume of 275 cubic centimeters.

101. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

102. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

103. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

104. The **signum function** is defined by

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\text{sgn}(x)$ and find the following (if possible).

(a) $\lim\limits_{x\to 0^-} \text{sgn}(x)$ (b) $\lim\limits_{x\to 0^+} \text{sgn}(x)$ (c) $\lim\limits_{x\to 0} \text{sgn}(x)$

105. ***Modeling Data*** After an object falls for t seconds, the spee S (in feet per second) of the object is recorded in the table.

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

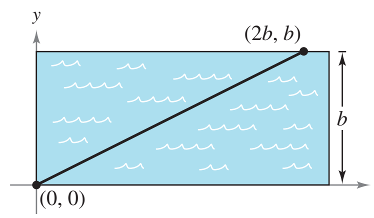
(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? I there is a limiting speed, identify a possible cause.

106. ***Creating Models*** A swimmer crosses a pool of width b b swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.

(a) Let f be a function defined as the y-coordinate of the poin on the long side of the pool that is nearest the swimmer a any given time during the swimmer's path across the pool Determine the function f and sketch its graph. Is i continuous? Explain.

(b) Let g be the minimum distance between the swimmer an the long sides of the pool. Determine the function g an sketch its graph. Is it continuous? Explain.



107. Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

108. Prove that for any real number y there exists x in $(-\pi/2, \pi/2$ such that $\tan x = y$.

109. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain o f? How can you define f at $x = 0$ in order for f to b continuous there?

110. Prove that if $\lim\limits_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuou at c.

111. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

112. (a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interva $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that ther exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. Us a graphing utility to approximate c to three decimal places

Putnam Exam Challenge

113. Prove or disprove: if x and y are real numbers with $y \ge 0$ an $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

114. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

These problems were composed by the Committee on the Putnam Prize Competition

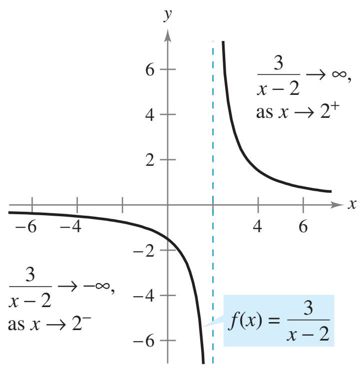
Section 1.5 Infinite Limits

- **Determine infinite limits from the left and from the right.**
- **Find and sketch the vertical asymptotes of the graph of a function.**

Infinite Limits

Let f be the function given by

$$f(x) = \frac{3}{x-2}.$$



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right. This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$$ $f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty$$ $f(x)$ increases without bound as x approaches 2 from the right.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	−6	−30	−300	−3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

Definition of Infinite Limits

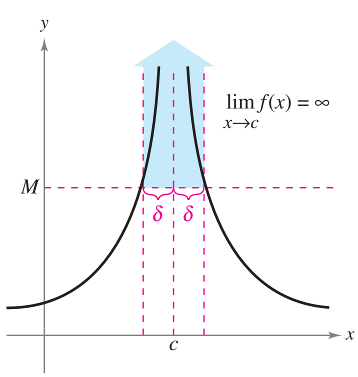
Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$. To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits
Figure 1.40

Video

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as x approaches c.

EXPLORATION

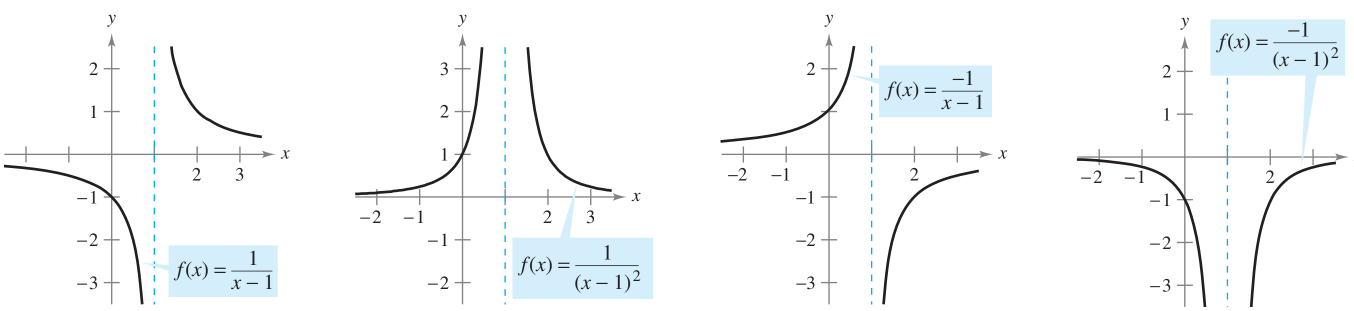
Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit of $f(x)$ as x approaches c from the left and from the right.

a. $f(x) = \dfrac{3}{x - 4}$ **b.** $f(x) = \dfrac{1}{2 - x}$

c. $f(x) = \dfrac{2}{(x - 3)^2}$ **d.** $f(x) = \dfrac{-3}{(x + 2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Use Figure 1.41 to determine the limit of each function as x approaches 1 from the left and from the right.



(a) (b) (c) (d)

Editable Graph Editable Graph Editable Graph Editable Graph

Figure 1.41 Each graph has an asymptote at $x = 1$.

Solution

a. $\displaystyle\lim_{x\to 1^-} \frac{1}{x - 1} = -\infty$ and $\displaystyle\lim_{x\to 1^+} \frac{1}{x - 1} = \infty$

b. $\displaystyle\lim_{x\to 1} \frac{1}{(x - 1)^2} = \infty$ Limit from each side is ∞.

c. $\displaystyle\lim_{x\to 1^-} \frac{-1}{x - 1} = \infty$ and $\displaystyle\lim_{x\to 1^+} \frac{-1}{x - 1} = -\infty$

d. $\displaystyle\lim_{x\to 1} \frac{-1}{(x - 1)^2} = -\infty$ Limit from each side is $-\infty$.

Try It Exploration A

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

NOTE If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number where the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation. (A proof of this theorem is given in Appendix A.)

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Video

EXAMPLE 2 Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \dfrac{1}{2(x+1)}$ **b.** $f(x) = \dfrac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

Solution

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x+1)}$$

is 0 and the numerator is not 0. So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.42(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Moreover, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.42(b).

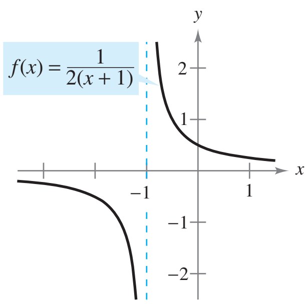
c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.42(c). So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur when $x = n\pi$, where n is an integer.

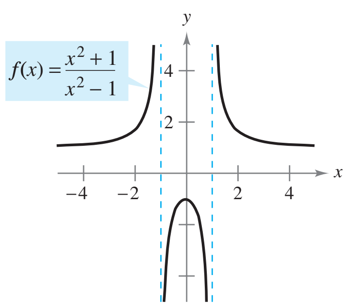
Try It | Exploration A | Exploration B | Open Exploration

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* 0/0, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.



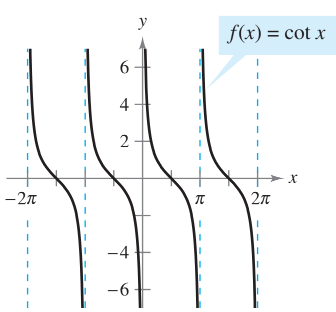
(a)

Editable Graph



(b)

Editable Graph



(c)

Editable Graph

Functions with vertical asymptotes
Figure 1.42

EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

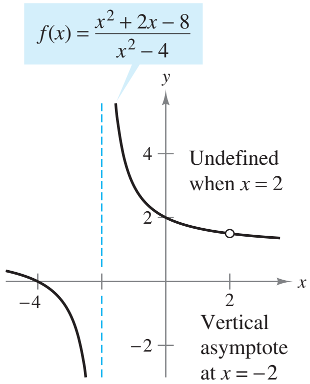
Solution Begin by simplifying the expression, as shown.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.43. From the graph, you can see that

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.43

Editable Graph

Try It | Exploration A | Exploration B

EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

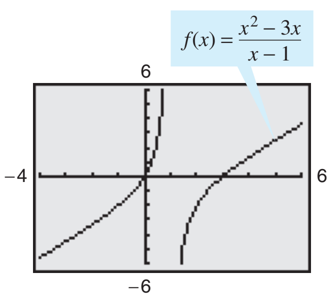
$$f(x) = \frac{x^2 - 3x}{x - 1}$$

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. A graphing utility can help determine the result. From the graph of f shown in Figure 1.44, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty \qquad \text{The limit from the left is infinity.}$$

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty. \qquad \text{The limit from the right is negative infinity.}$$



f has a vertical asymptote at $x = 1$.
Figure 1.44

Editable Graph

Try It | Exploration A

TECHNOLOGY PITFALL When using a graphing calculator or graphing software, be careful to interpret correctly the graph of a function with a vertical asymptote—graphing utilities often have difficulty drawing this type of graph.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Proof To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that

$$[f(x) + g(x)] > M$$

whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive. Let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

The proofs of the remaining properties are left as exercises (see Exercise 72).

EXAMPLE 5 Determining Limits

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0} \left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0. \qquad \text{Property 3, Theorem 1.15}$$

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

Try It

Exploration A

Exercises for Section 1.5

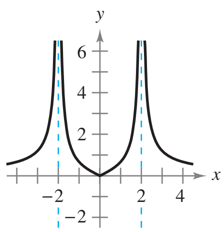
The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

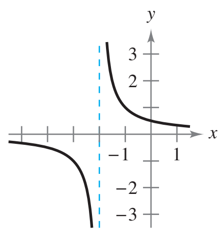
Click on M to print an enlarged copy of the graph.

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

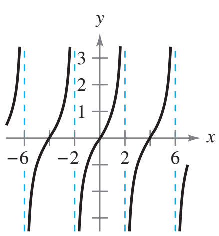
1. $f(x) = 2\left|\dfrac{x}{x^2 - 4}\right|$



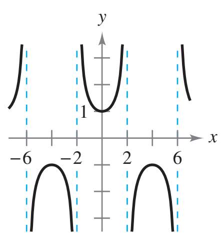
2. $f(x) = \dfrac{1}{x + 2}$



3. $f(x) = \tan \dfrac{\pi x}{4}$



4. $f(x) = \sec \dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function and confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

5. $f(x) = \dfrac{1}{x^2 - 9}$

6. $f(x) = \dfrac{x}{x^2 - 9}$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

8. $f(x) = \sec \dfrac{\pi x}{6}$

In Exercises 9–28, find the vertical asymptotes (if any) of the graph of the function.

9. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = \dfrac{4}{(x - 2)^3}$

11. $h(x) = \dfrac{x^2 - 2}{x^2 - x - 2}$

12. $g(x) = \dfrac{2 + x}{x^2(1 - x)}$

13. $f(x) = \dfrac{x^2}{x^2 - 4}$

14. $f(x) = \dfrac{-4x}{x^2 + 4}$

15. $g(t) = \dfrac{t - 1}{t^2 + 1}$

16. $h(s) = \dfrac{2s - 3}{s^2 - 25}$

17. $f(x) = \tan 2x$

18. $f(x) = \sec \pi x$

19. $T(t) = 1 - \dfrac{4}{t^2}$

20. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2 - 6x - 24}$

21. $f(x) = \dfrac{x}{x^2 + x - 2}$

22. $f(x) = \dfrac{4x^2 + 4x - 24}{x^4 - 2x^3 - 9x^2 + 18x}$

23. $g(x) = \dfrac{x^3 + 1}{x + 1}$

24. $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2}$

25. $f(x) = \dfrac{x^2 - 2x - 15}{x^3 - 5x^2 + x - 5}$

26. $h(t) = \dfrac{t^2 - 2t}{t^4 - 16}$

27. $s(t) = \dfrac{t}{\sin t}$

28. $g(\theta) = \dfrac{\tan \theta}{\theta}$

In Exercises 29–32, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

29. $f(x) = \dfrac{x^2 - 1}{x + 1}$

30. $f(x) = \dfrac{x^2 - 6x - 7}{x + 1}$

31. $f(x) = \dfrac{x^2 + 1}{x + 1}$

32. $f(x) = \dfrac{\sin(x + 1)}{x + 1}$

In Exercises 33–48, find the limit.

33. $\displaystyle\lim_{x \to 2^+} \frac{x - 3}{x - 2}$

34. $\displaystyle\lim_{x \to 1^+} \frac{2 + x}{1 - x}$

35. $\displaystyle\lim_{x \to 3^+} \frac{x^2}{x^2 - 9}$

36. $\displaystyle\lim_{x \to 4^-} \frac{x^2}{x^2 + 16}$

37. $\displaystyle\lim_{x \to -3^-} \frac{x^2 + 2x - 3}{x^2 + x - 6}$

38. $\displaystyle\lim_{x \to (-1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3}$

39. $\displaystyle\lim_{x \to 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)}$

40. $\displaystyle\lim_{x \to 3} \frac{x - 2}{x^2}$

41. $\displaystyle\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right)$

42. $\displaystyle\lim_{x \to 0^-} \left(x^2 - \frac{1}{x}\right)$

43. $\displaystyle\lim_{x \to 0^+} \frac{2}{\sin x}$

44. $\displaystyle\lim_{x \to (\pi/2)^+} \frac{-2}{\cos x}$

45. $\displaystyle\lim_{x \to \pi} \frac{\sqrt{x}}{\csc x}$

46. $\displaystyle\lim_{x \to 0} \frac{x + 2}{\cot x}$

47. $\displaystyle\lim_{x \to 1/2} x \sec \pi x$

48. $\displaystyle\lim_{x \to 1/2} x^2 \tan \pi x$

In Exercises 49–52, use a graphing utility to graph the function and determine the one-sided limit.

49. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\displaystyle\lim_{x \to 1^+} f(x)$

50. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\displaystyle\lim_{x \to 1^-} f(x)$

51. $f(x) = \dfrac{1}{x^2 - 25}$

$\displaystyle\lim_{x \to 5^-} f(x)$

52. $f(x) = \sec \dfrac{\pi x}{6}$

$\displaystyle\lim_{x \to 3^+} f(x)$

Writing About Concepts

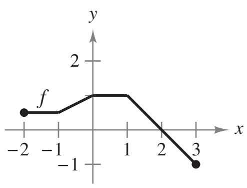
53. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

54. In your own words, describe what is meant by an asymptote of a graph.

55. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Does the graph of every rational function have a vertical asymptote? Explain.

57. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, select the MathGraph button.



58. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

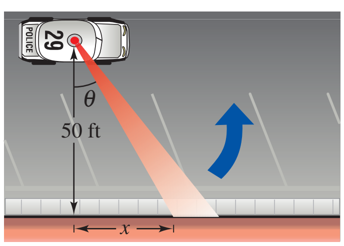
59. ***Rate of Change*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is

$$r = 50\pi \sec^2 \theta \text{ ft/sec}.$$

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.



60. ***Illegal Drugs*** The cost in millions of dollars for a governmental agency to seize $x\%$ of an illegal drug is

$$C = \frac{528x}{100 - x}, \quad 0 \le x < 100.$$

(a) Find the cost of seizing 25% of the drug.

(b) Find the cost of seizing 50% of the drug.

(c) Find the cost of seizing 75% of the drug.

(d) Find the limit of C as $x \to 100^-$ and interpret its meaning.

61. ***Relativity*** According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

62. ***Rate of Change*** A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

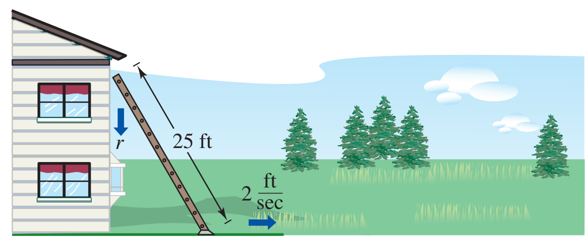
$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.



63. ***Average Speed*** On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as $x \to 25^+$ and interpret its meaning.

64. ***Numerical and Graphical Analysis*** Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power on x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

(a) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x}$ (b) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^2}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^3}$ (d) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^4}$

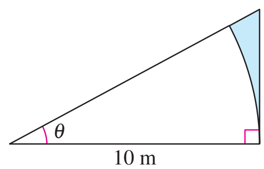
65. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).

(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

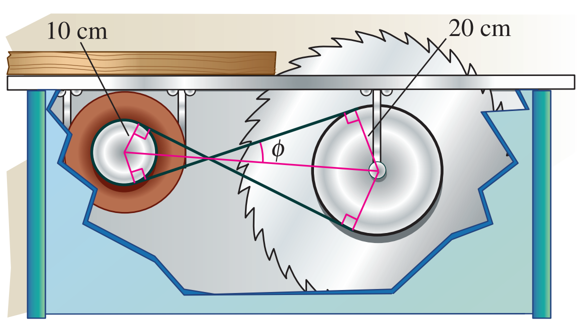
(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as $\theta \to (\pi/2)^-$.



66. ***Numerical and Graphical Reasoning*** A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appro priate domain.

(f) Find $\lim_{\phi \to (\pi/2)^-} L$. Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.

True or False? **In Exercises 67–70, determine whether th statement is true or false. If it is false, explain why or give a example that shows it is false.**

67. If $p(x)$ is a polynomial, then the graph of the function given by $f(x) = \dfrac{p(x)}{x-1}$ has a vertical asymptote at $x = 1$.

68. The graph of a rational function has at least one vertica asymptote.

69. The graphs of polynomial functions have no vertica asymptotes.

70. If f has a vertical asymptote at $x = 0$, then f is undefined a $x = 0$.

71. Find functions f and g such that $\lim_{x \to c} f(x) = \infty$ an $\lim_{x \to c} g(x) = \infty$ but $\lim_{x \to c} [f(x) - g(x)] \neq 0$.

72. Prove the remaining properties of Theorem 1.15.

73. Prove that if $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} \dfrac{1}{f(x)} = 0$.

74. Prove that if $\lim_{x \to c} \dfrac{1}{f(x)} = 0$, then $\lim_{x \to c} f(x)$ does not exist.

Infinite Limits **In Exercises 75 and 76, use the ε-δ definition o infinite limits to prove the statement.**

75. $\lim_{x \to 3^+} \dfrac{1}{x-3} = \infty$

76. $\lim_{x \to 4^-} \dfrac{1}{x-4} = -\infty$

Review Exercises for Chapter 1

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1 and 2, determine whether the problem can be solved using precalculus or if calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning. Use a graphical or numerical approach to estimate the solution.

1. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the curve $y = x^2$.
2. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the line $y = 4x - 3$.

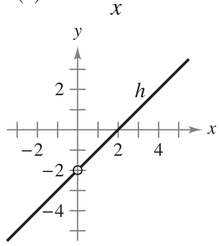
In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\lim_{x\to 0} \dfrac{[4/(x+2)] - 2}{x}$
4. $\lim_{x\to 0} \dfrac{4\left(\sqrt{x+2} - \sqrt{2}\right)}{x}$

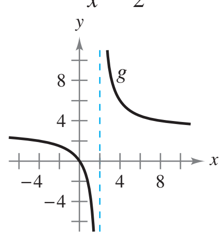
In Exercises 5 and 6, use the graph to determine each limit.

5. $h(x) = \dfrac{x^2 - 2x}{x}$



(a) $\lim_{x\to 0} h(x)$ (b) $\lim_{x\to -1} h(x)$

6. $g(x) = \dfrac{3x}{x - 2}$



(a) $\lim_{x\to 2} g(x)$ (b) $\lim_{x\to 0} g(x)$

In Exercises 7–10, find the limit L. Then use the ε-δ definition to prove that the limit is L.

7. $\lim_{x\to 1} (3 - x)$
8. $\lim_{x\to 9} \sqrt{x}$
9. $\lim_{x\to 2} (x^2 - 3)$
10. $\lim_{x\to 5} 9$

In Exercises 11–24, find the limit (if it exists).

11. $\lim_{t\to 4} \sqrt{t+2}$
12. $\lim_{y\to 4} 3|y - 1|$
13. $\lim_{t\to -2} \dfrac{t+2}{t^2 - 4}$
14. $\lim_{t\to 3} \dfrac{t^2 - 9}{t - 3}$
15. $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$
16. $\lim_{x\to 0} \dfrac{\sqrt{4+x} - 2}{x}$
17. $\lim_{x\to 0} \dfrac{[1/(x+1)] - 1}{x}$
18. $\lim_{s\to 0} \dfrac{\left(1/\sqrt{1+s}\right) - 1}{s}$
19. $\lim_{x\to -5} \dfrac{x^3 + 125}{x + 5}$
20. $\lim_{x\to -2} \dfrac{x^2 - 4}{x^3 + 8}$
21. $\lim_{x\to 0} \dfrac{1 - \cos x}{\sin x}$
22. $\lim_{x\to \pi/4} \dfrac{4x}{\tan x}$
23. $\lim_{\Delta x\to 0} \dfrac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x}$

    [*Hint:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$]
24. $\lim_{\Delta x\to 0} \dfrac{\cos(\pi + \Delta x) + 1}{\Delta x}$

    [*Hint:* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$]

In Exercises 25 and 26, evaluate the limit given $\lim_{x\to c} f(x) = -$ and $\lim_{x\to c} g(x) = \frac{2}{3}$.

25. $\lim_{x\to c} [f(x)g(x)]$
26. $\lim_{x\to c} [f(x) + 2g(x)]$

Numerical, Graphical, and Analytic Analysis **In Exercises 2 and 28, consider**

$$\lim_{x\to 1^+} f(x).$$

(a) Complete the table to estimate the limit.

(b) Use a graphing utility to graph the function and use th graph to estimate the limit.

(c) Rationalize the numerator to find the exact value of th limit analytically.

x	1.1	1.01	1.001	1.0001
$f(x)$				

27. $f(x) = \dfrac{\sqrt{2x+1} - \sqrt{3}}{x - 1}$
28. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x - 1}$

    [*Hint:* $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Free-Falling Object **In Exercises 29 and 30, use the positio function $s(t) = -4.9t^2 + 200$, which gives the height (in meters of an object that has fallen from a height of 200 meters. Th velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

29. Find the velocity of the object when $t = 4$.
30. At what velocity will the object impact the ground?

In Exercises 31–36, find the limit (if it exists). If the limit does not exist, explain why.

31. $\displaystyle\lim_{x\to 3^-} \frac{|x-3|}{x-3}$

32. $\displaystyle\lim_{x\to 4} [\![x-1]\!]$

33. $\displaystyle\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

34. $\displaystyle\lim_{x\to 1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

35. $\displaystyle\lim_{t\to 1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

36. $\displaystyle\lim_{s\to -2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

In Exercises 37–46, determine the intervals on which the function is continuous.

37. $f(x) = [\![x+3]\!]$

38. $f(x) = \dfrac{3x^2-x-2}{x-1}$

39. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

40. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

41. $f(x) = \dfrac{1}{(x-2)^2}$

42. $f(x) = \sqrt{\dfrac{x+1}{x}}$

43. $f(x) = \dfrac{3}{x+1}$

44. $f(x) = \dfrac{x+1}{2x+2}$

45. $f(x) = \csc \dfrac{\pi x}{2}$

46. $f(x) = \tan 2x$

47. Determine the value of c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

48. Determine the values of b and c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

49. Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

50. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$9.80 for the first pound and \$2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Use a graphing utility to graph the function and discuss its continuity.

51. Let $f(x) = \dfrac{x^2-4}{|x-2|}$. Find each limit (if possible).

(a) $\displaystyle\lim_{x\to 2^-} f(x)$

(b) $\displaystyle\lim_{x\to 2^+} f(x)$

(c) $\displaystyle\lim_{x\to 2} f(x)$

52. Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\displaystyle\lim_{x\to 0^-} f(x)$.

(c) Find $\displaystyle\lim_{x\to 1^+} f(x)$.

In Exercises 53–56, find the vertical asymptotes (if any) of the graphs of the function.

53. $g(x) = 1 + \dfrac{2}{x}$

54. $h(x) = \dfrac{4x}{4-x^2}$

55. $f(x) = \dfrac{8}{(x-10)^2}$

56. $f(x) = \csc \pi x$

In Exercises 57–68, find the one-sided limit.

57. $\displaystyle\lim_{x\to -2^-} \frac{2x^2+x+1}{x+2}$

58. $\displaystyle\lim_{x\to (1/2)^+} \frac{x}{2x-1}$

59. $\displaystyle\lim_{x\to -1^+} \frac{x+1}{x^3+1}$

60. $\displaystyle\lim_{x\to -1^-} \frac{x+1}{x^4-1}$

61. $\displaystyle\lim_{x\to 1^-} \frac{x^2+2x+1}{x-1}$

62. $\displaystyle\lim_{x\to -1^+} \frac{x^2-2x+1}{x+1}$

63. $\displaystyle\lim_{x\to 0^+} \left(x - \frac{1}{x^3}\right)$

64. $\displaystyle\lim_{x\to 2^-} \frac{1}{\sqrt[3]{x^2-4}}$

65. $\displaystyle\lim_{x\to 0^+} \frac{\sin 4x}{5x}$

66. $\displaystyle\lim_{x\to 0^+} \frac{\sec x}{x}$

67. $\displaystyle\lim_{x\to 0^+} \frac{\csc 2x}{x}$

68. $\displaystyle\lim_{x\to 0^-} \frac{\cos^2 x}{x}$

69. ***Environment*** A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \qquad 0 \le p < 100.$$

Find the cost of removing (a) 15%, (b) 50%, and (c) 90% of the pollutants. (d) Find the limit of C as $p \to 100^-$.

70. The function f is defined as shown.

$$f(x) = \frac{\tan 2x}{x}, \qquad x \ne 0$$

(a) Find $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

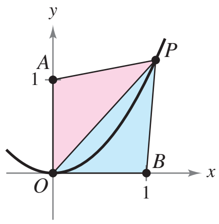
P.S. Problem Solving

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on **S** to view the complete solution of the exercise.

Click on **M** to print an enlarged copy of the graph.

1. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the perimeter of each triangle in terms of x.

(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

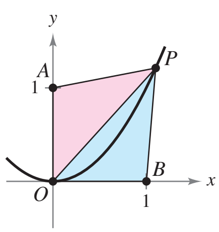
$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

(c) Calculate $\lim_{x\to 0^+} r(x)$.

2. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

(c) Calculate $\lim_{x\to 0^+} a(x)$.

3. (a) Find the area of a regular hexagon inscribed in a circle of radius 1. How close is this area to that of the circle?

(b) Find the area A_n of an n-sided regular polygon inscribed in a circle of radius 1. Write your answer as a function of n.

(c) Complete the table.

n	6	12	24	48	96
A_n					

(d) What number does A_n approach as n gets larger and larger?

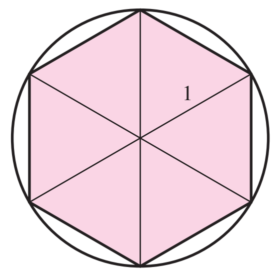


Figure for 3

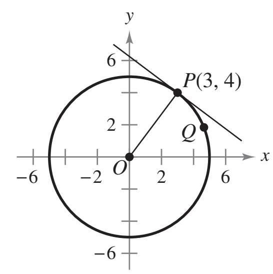


Figure for 4

4. Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

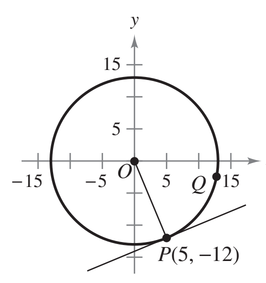
(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x\to 3} m_x$. How does this number relate to your answer in part (b)?

5. Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.



(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x\to 5} m_x$. How does this number relate to your answer in part (b)?

6. Find the values of the constants a and b such that

$$\lim_{x\to 0} \frac{\sqrt{a + bx} - \sqrt{3}}{x} = \sqrt{3}.$$

7. Consider the function $f(x) = \dfrac{\sqrt{3 + x^{1/3}} - 2}{x - 1}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

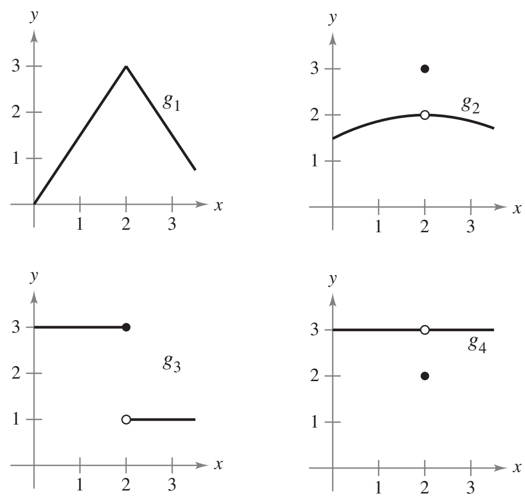
(c) Calculate $\displaystyle\lim_{x \to -27^+} f(x)$.

(d) Calculate $\displaystyle\lim_{x \to 1} f(x)$.

8. Determine all values of the constant a such that the following function is continuous for all real numbers.

$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \geq 0 \\ a^2 - 2, & x < 0 \end{cases}$$

9. Consider the graphs of the four functions g_1, g_2, g_3, and g_4.



For each given condition of the function f, which of the graphs could be the graph of f?

(a) $\displaystyle\lim_{x \to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\displaystyle\lim_{x \to 2^-} f(x) = 3$

10. Sketch the graph of the function $f(x) = \left[\!\left[\dfrac{1}{x} \right]\!\right]$.

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\displaystyle\lim_{x \to 1^-} f(x)$, $\displaystyle\lim_{x \to 1^+} f(x)$, $\displaystyle\lim_{x \to 0^-} f(x)$, and $\displaystyle\lim_{x \to 0^+} f(x)$.

(c) Discuss the continuity of the function.

11. Sketch the graph of the function $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\displaystyle\lim_{x \to 1^-} f(x)$, $\displaystyle\lim_{x \to 1^+} f(x)$, and $\displaystyle\lim_{x \to \frac{1}{2}} f(x)$.

(c) Discuss the continuity of the function.

12. To escape Earth's gravitational field, a rocket must be launche with an initial velocity called the **escape velocity.** A rocke launched from the surface of Earth has velocity v (in miles pe second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket t the center of Earth, G is the gravitational constant, M is the mas of Earth, and R is the radius of Earth (approximately 4000 miles)

(a) Find the value of v_0 for which you obtain an infinite limi for r as v tends to zero. This value of v_0 is the escap velocity for Earth.

(b) A rocket launched from the surface of the moon ha velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a planet has velocit v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of thi planet larger or smaller than that of Earth? (Assume that th mean density of this planet is the same as that of Earth.)

13. For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x - a) - H(x - b) = \begin{cases} 0, & x < a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\displaystyle\lim_{x \to a^+} P_{a,b}(x)$ (ii) $\displaystyle\lim_{x \to a^-} P_{a,b}(x)$

(iii) $\displaystyle\lim_{x \to b^+} P_{a,b}(x)$ (iv) $\displaystyle\lim_{x \to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is

$$U(x) = \frac{1}{b - a} P_{a,b}(x)$$

called the **unit** pulse function?

14. Let a be a nonzero constant. Prove that if $\displaystyle\lim_{x \to 0} f(x) = L$, the $\displaystyle\lim_{x \to 0} f(ax) = L$. Show by means of an example that a must b nonzero.

Section 2.1 The Derivative and the Tangent Line Problem

- **Find the slope of the tangent line to a curve at a point.**
- **Use the limit definition to find the derivative of a function.**
- **Understand the relationship between differentiability and continuity.**

ISAAC NEWTON (1642–1727)
In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Law of Universal Gravitation and his three laws of motion.

MathBio

The Tangent Line Problem

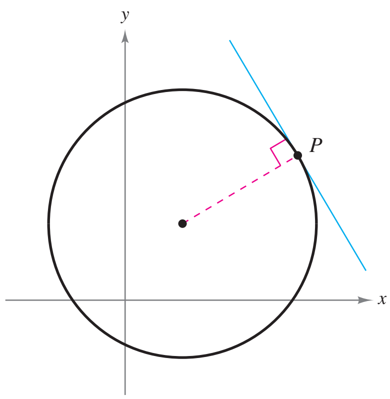
Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

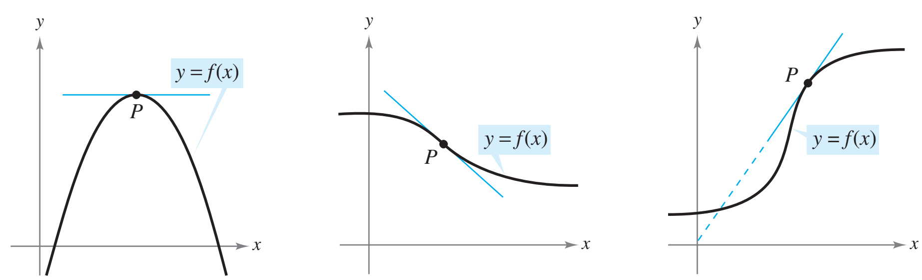
A brief introduction to the tangent line problem is given in Section 1.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at point P, as shown in Figure 2.1.



Tangent line to a circle
Figure 2.1

For a general curve, however, the problem is more difficult. For example, how would you define the tangent lines shown in Figure 2.2? You might say that a line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in Figure 2.2, but not for the second. *Or* you might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point. This definition would work for a circle but not for more general curves, as the third curve in Figure 2.2 shows.



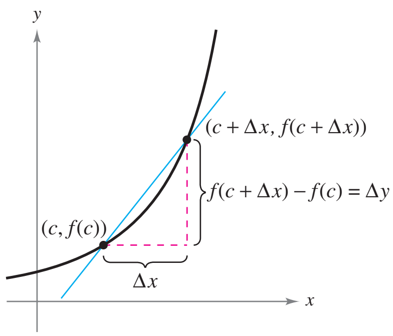
Tangent line to a curve at a point
Figure 2.2

FOR FURTHER INFORMATION For more information on the crediting of mathematical discoveries to the first "discoverer," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*.

MathArticle

EXPLORATION

Identifying a Tangent Line Use a graphing utility to graph the function $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, graph $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of f at the point $(0, -5)$? Explain your reasoning.



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$
Figure 2.3

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the *slope* of the tangent line at point P. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 2.3. If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph of f, the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \text{Slope of secant line}$$

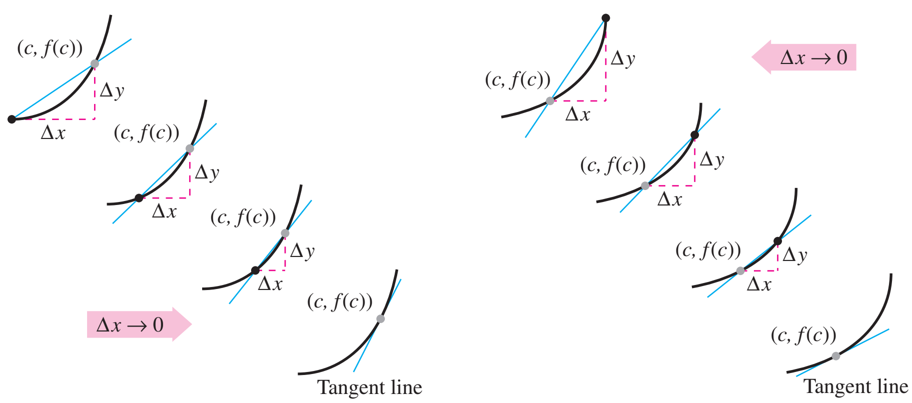
The right-hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x,** and the numerator $\Delta y = f(c + \Delta x) - f(c)$ is the **change in y.**

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.

The Tangent Line Problem

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."



Tangent line approximations
Figure 2.4

To view a sequence of secant lines approaching a tangent line, select the Animation button.

Animation

Definition of Tangent Line with Slope m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Video Video

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

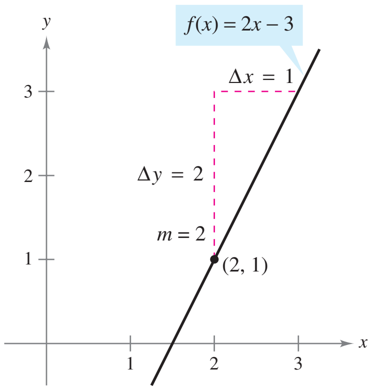
* *This use of the word* secant *comes from the Latin* secare, *meaning to cut, and is not a reference to the trigonometric function of the same name.*

EXAMPLE 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point (2, 1).



The slope of f at (2, 1) is $m = 2$.
Figure 2.5

Editable Graph

Solution To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2.5.

NOTE In Example 1, the limit definition of the slope of f agrees with the definition of the slope of a line as discussed in Section P.2.

Try It Exploration A

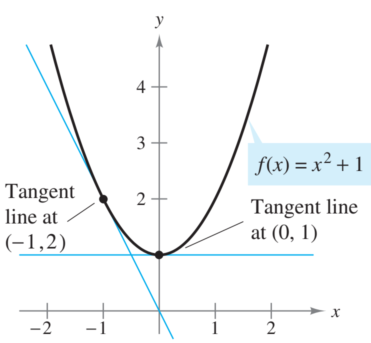
The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the following example.

EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

at the points (0, 1) and $(-1, 2)$, as shown in Figure 2.6.



The slope of f at any point $(c, f(c))$ is $m = 2c$.
Figure 2.6

Editable Graph

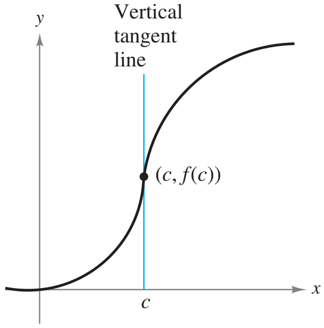
Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2c + \Delta x) \\
&= 2c.
\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point (0, 1), the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$.

NOTE In Example 2, note that c is held constant in the limit process (as $\Delta x \to 0$).

Try It Exploration A



The graph of f has a vertical tangent line at $(c, f(c))$.
Figure 2.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

Definition of the Derivative of a Function

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Video

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y]. \qquad \text{Notation for derivatives}$$

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "$dy - dx$". Using limit notation, you can write

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x).\end{aligned}$$

History

EXAMPLE 3 Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3 + 2x$.

Solution

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2] \\ &= 3x^2 + 2 \end{aligned}$$

STUDY TIP When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

Try It | Exploration A | Exploration B | Exploration C | Open Exploration

The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

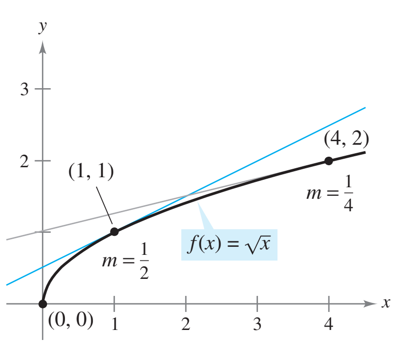
Remember that the derivative of a function f is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slope of the graph of f at the points (1, 1) and (4, 2). Discuss the behavior of f at (0, 0).

Solution Use the procedure for rationalizing numerators, as discussed in Section 1.3.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right) \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}, \quad x > 0 \end{aligned}$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/\left(2\sqrt{x}\right)$.
Figure 2.8

Editable Graph

At the point (1, 1), the slope is $f'(1) = \frac{1}{2}$. At the point (4, 2), the slope is $f'(4) = \frac{1}{4}$. See Figure 2.8. At the point (0, 0), the slope is undefined. Moreover, the graph of f has a vertical tangent line at (0, 0).

Try It | Exploration A | Exploration B | Exploration C

In many applications, it is convenient to use a variable other than x as the independent variable, as shown in Example 5.

EXAMPLE 5 Finding the Derivative of a Function

Find the derivative with respect to t for the function $y = 2/t$.

Solution Considering $y = f(t)$, you obtain

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} \qquad f(t + \Delta t) = 2/(t + \Delta t) \text{ and } f(t) = 2/t$$

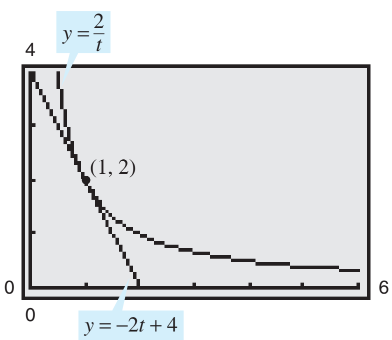
$$= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} \qquad \text{Combine fractions in numerator.}$$

$$= \lim_{\Delta t \to 0} \frac{-2\Delta t}{\Delta t(t)(t + \Delta t)} \qquad \text{Divide out common factor of } \Delta t.$$

$$= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} \qquad \text{Simplify.}$$

$$= -\frac{2}{t^2}. \qquad \text{Evaluate limit as } \Delta t \to 0.$$

Try It   Exploration A   Open Exploration



At the point $(1, 2)$ the line $y = -2t + 4$ is tangent to the graph of $y = 2/t$.
Figure 2.9

The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

TECHNOLOGY A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. This implies that an equation of the tangent line to the graph at $(1, 2)$ is

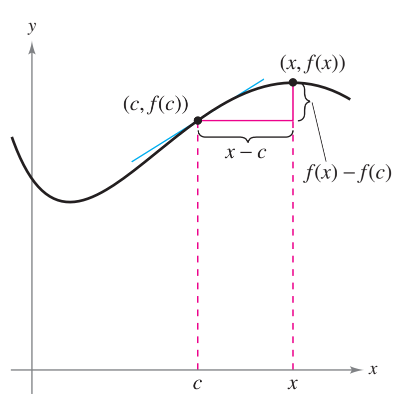
$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 2.9.

Differentiability and Continuity

The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

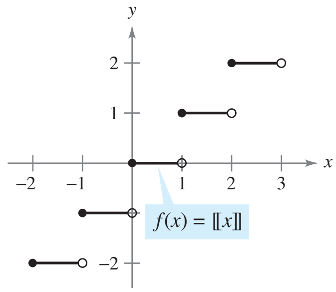


As x approaches c, the secant line approaches the tangent line.
Figure 2.10

provided this limit exists (see Figure 2.10). (A proof of the equivalence of this form is given in Appendix A.) Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval** $[a, b]$ if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.



The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.
Figure 2.11

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function

$$f(x) = [\![x]\!]$$

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 2.11). You can verify this by observing that

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{[\![x]\!] - 0}{x} = \infty \qquad \text{Derivative from the left}$$

and

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{[\![x]\!] - 0}{x} = 0. \qquad \text{Derivative from the right}$$

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate this possibility.

EXAMPLE 6 A Graph with a Sharp Turn

The function

$$f(x) = |x - 2|$$

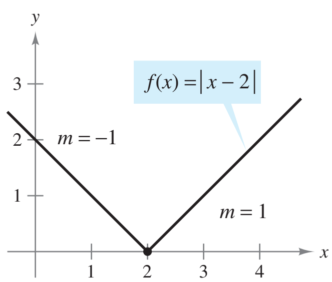
shown in Figure 2.12 is continuous at $x = 2$. But, the one-sided limits

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \qquad \text{Derivative from the left}$$

and

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \qquad \text{Derivative from the right}$$

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point $(2, 0)$.



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.
Figure 2.12

Editable Graph

Try It | Exploration A | Open Exploration

EXAMPLE 7 A Graph with a Vertical Tangent Line

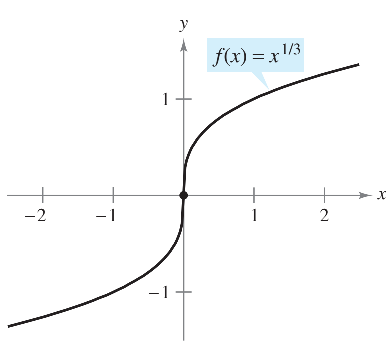
The function

$$f(x) = x^{1/3}$$

is continuous at $x = 0$, as shown in Figure 2.13. But, because the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^{1/3} - 0}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$.



f is not differentiable at $x = 0$, because f has a vertical tangent at $x = 0$.
Figure 2.13

Editable Graph

Try It | Exploration A | Exploration B | Exploration C

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent.

TECHNOLOGY Some graphing utilities, such as *Derive*, *Maple*, *Mathcad*, *Mathematica*, and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}
\lim_{x\to c}[f(x) - f(c)] &= \lim_{x\to c}\left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\
&= \left[\lim_{x\to c}(x - c)\right]\left[\lim_{x\to c}\frac{f(x) - f(c)}{x - c}\right] \\
&= (0)[f'(c)] \\
&= 0
\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. So, f is continuous at $x = c$.

The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

Exercises for Section 2.1

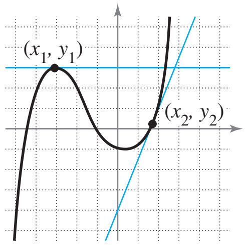
The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

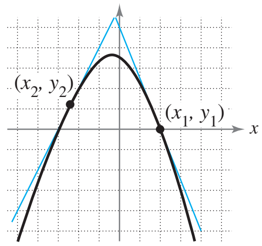
Click on M to print an enlarged copy of the graph.

In Exercises 1 and 2, estimate the slope of the graph at the points (x_1, y_1) and (x_2, y_2).

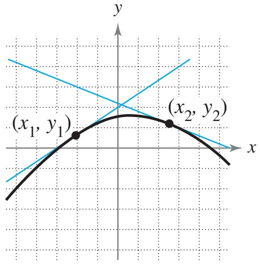
1. (a)



(b)



2. (a)



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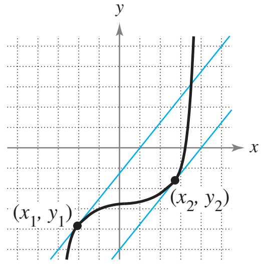
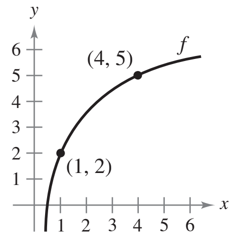


In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, select the MathGraph button.



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol (< or >) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad \dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

5. $f(x) = 3 - 2x, \quad (-1, 5)$

6. $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

7. $g(x) = x^2 - 4, \quad (1, -3)$

8. $g(x) = 5 - x^2, \quad (2, 1)$

9. $f(t) = 3t - t^2, \quad (0, 0)$

10. $h(t) = t^2 + 3, \quad (-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11. $f(x) = 3$

12. $g(x) = -5$

13. $f(x) = -5x$

14. $f(x) = 3x + 2$

15. $h(s) = 3 + \frac{2}{3}s$

16. $f(x) = 9 - \frac{1}{2}x$

17. $f(x) = 2x^2 + x - 1$

18. $f(x) = 1 - x^2$

19. $f(x) = x^3 - 12x$

20. $f(x) = x^3 + x^2$

21. $f(x) = \dfrac{1}{x - 1}$

22. $f(x) = \dfrac{1}{x^2}$

23. $f(x) = \sqrt{x + 1}$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

In Exercises 25–32, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 1, \quad (2, 5)$

26. $f(x) = x^2 + 2x + 1, \quad (-3, 4)$

27. $f(x) = x^3, \quad (2, 8)$

28. $f(x) = x^3 + 1, \quad (1, 2)$

29. $f(x) = \sqrt{x}, \quad (1, 1)$

30. $f(x) = \sqrt{x - 1}, \quad (5, 2)$

31. $f(x) = x + \dfrac{4}{x}, \quad (4, 5)$

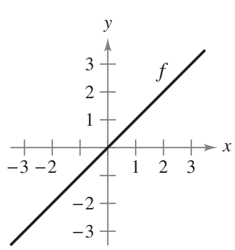
32. $f(x) = \dfrac{1}{x + 1}, \quad (0, 1)$

In Exercises 33–36, find an equation of the line that is tangent to the graph of *f and* parallel to the given line.

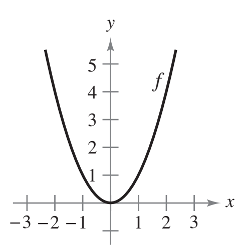
	Function	*Line*
33.	$f(x) = x^3$	$3x - y + 1 = 0$
34.	$f(x) = x^3 + 2$	$3x - y - 4 = 0$
35.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
36.	$f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

In Exercises 37–40, the graph of *f* is given. Select the graph of *f′*.

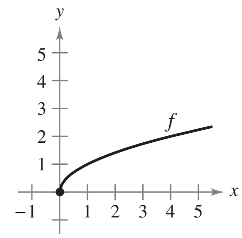
37.



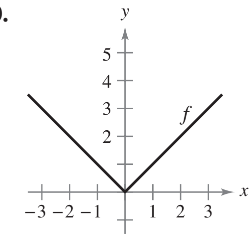
38.



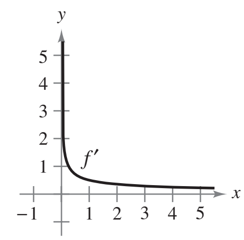
39.



40.



(a)



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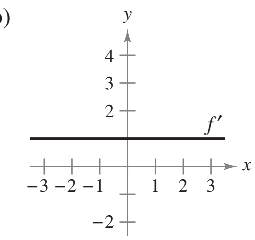

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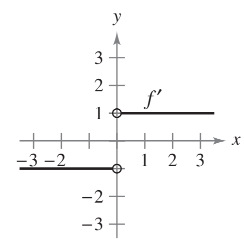

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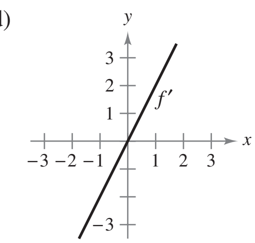


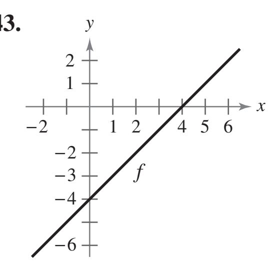
41. The tangent line to the graph of $y = g(x)$ at the point $(5, 2)$ passes through the point $(9, 0)$. Find $g(5)$ and $g'(5)$.

42. The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

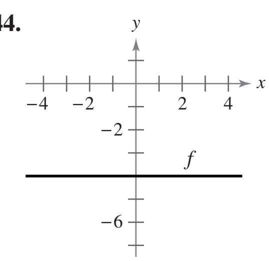
Writing About Concepts

In Exercises 43–46, sketch the graph of *f′*. Explain how you found your answer.

43.



44.



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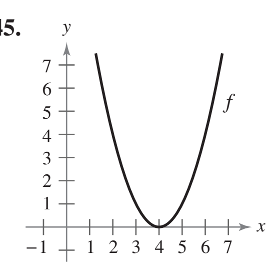
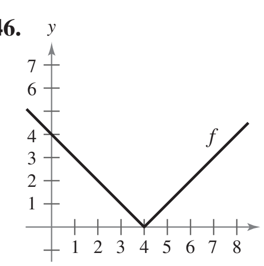


46.



47. Sketch a graph of a function whose derivative is always negative.

Writing About Concepts *(continued)*

48. Sketch a graph of a function whose derivative is always positive.

In Exercises 49–52, the limit represents $f'(c)$ for a function f and a number c. Find f and c.

49. $\lim_{\Delta x \to 0} \frac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$ **50.** $\lim_{\Delta x \to 0} \frac{(-2 + \Delta x)^3 + 8}{\Delta x}$

51. $\lim_{x \to 6} \frac{-x^2 + 36}{x - 6}$ **52.** $\lim_{x \to 9} \frac{2\sqrt{x} - 6}{x - 9}$

In Exercises 53–55, identify a function f that has the following characteristics. Then sketch the function.

53. $f(0) = 2;$ $f'(x) = -3, -\infty < x < \infty$

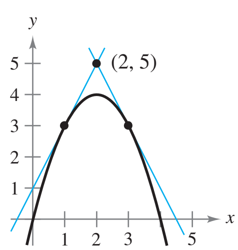
54. $f(0) = 4; f'(0) = 0;$ $f'(x) < 0$ for $x < 0;$ $f'(x) > 0$ for $x > 0$

55. $f(0) = 0; f'(0) = 0; f'(x) > 0$ if $x \neq 0$

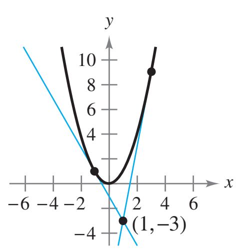
56. Assume that $f'(c) = 3$. Find $f'(-c)$ if (a) f is an odd function and if (b) f is an even function.

In Exercises 57 and 58, find equations of the two tangent lines to the graph of f that pass through the indicated point.

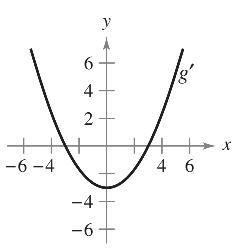
57. $f(x) = 4x - x^2$



58. $f(x) = x^2$



59. ***Graphical Reasoning*** The figure shows the graph of g'.



(a) $g'(0) =$ ▭ (b) $g'(3) =$ ▭

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

60. ***Graphical Reasoning*** Use a graphing utility to graph eacl function and its tangent lines at $x = -1$, $x = 0$, and $x = 1$ Based on the results, determine whether the slopes of tangen lines to the graph of a function at different values of x ar always distinct.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

Graphical, Numerical, and Analytic Analysis **In Exercises 6: and 62, use a graphing utility to graph f on the interval $[-2, 2]$ Complete the table by graphically estimating the slopes of th graph at the indicated points. Then evaluate the slopes analyti cally and compare your results with those obtained graphically**

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

61. $f(x) = \frac{1}{4}x^3$ **62.** $f(x) = \frac{1}{2}x^2$

Graphical Reasoning **In Exercises 63 and 64, use a graphin utility to graph the functions f and g in the same viewin window where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

63. $f(x) = 2x - x^2$ **64.** $f(x) = 3\sqrt{x}$

In Exercises 65 and 66, evaluate $f(2)$ and $f(2.1)$ and use th results to approximate $f'(2)$.

65. $f(x) = x(4 - x)$ **66.** $f(x) = \frac{1}{4}x^3$

Graphical Reasoning **In Exercises 67 and 68, use a graphin utility to graph the function and its derivative in the sam viewing window. Label the graphs and describe the relationshi between them.**

67. $f(x) = \frac{1}{\sqrt{x}}$ **68.** $f(x) = \frac{x^3}{4} - 3x$

Writing **In Exercises 69 and 70, consider the functions f an $S_{\Delta x}$ where**

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2).$$

(a) Use a graphing utility to graph f and $S_{\Delta x}$ in the sam viewing window for $\Delta x = 1, 0.5,$ and 0.1.

(b) Give a written description of the graphs of S for the differen values of Δx in part (a).

69. $f(x) = 4 - (x - 3)^2$ **70.** $f(x) = x + \frac{1}{x}$

In Exercises 71–80, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

71. $f(x) = x^2 - 1,\quad c = 2$

72. $g(x) = x(x - 1),\quad c = 1$

73. $f(x) = x^3 + 2x^2 + 1,\quad c = -2$

74. $f(x) = x^3 + 2x,\quad c = 1$

75. $g(x) = \sqrt{|x|},\quad c = 0$

76. $f(x) = 1/x,\quad c = 3$

77. $f(x) = (x - 6)^{2/3},\quad c = 6$

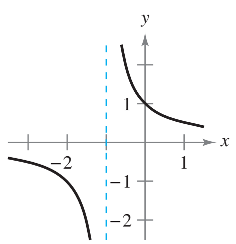
78. $g(x) = (x + 3)^{1/3},\quad c = -3$

79. $h(x) = |x + 5|,\quad c = -5$

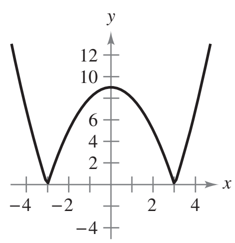
80. $f(x) = |x - 4|,\quad c = 4$

In Exercises 81–86, describe the x-values at which f is differentiable.

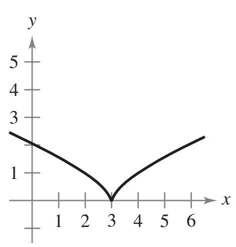
81. $f(x) = \dfrac{1}{x + 1}$



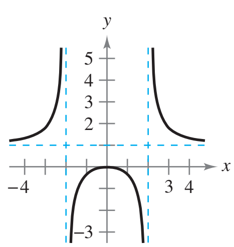
82. $f(x) = |x^2 - 9|$



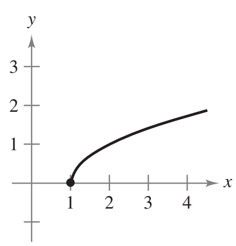
83. $f(x) = (x - 3)^{2/3}$



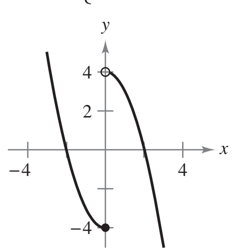
84. $f(x) = \dfrac{x^2}{x^2 - 4}$



85. $f(x) = \sqrt{x - 1}$



86. $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$



Graphical Analysis **In Exercises 87–90, use a graphing utility to find the x-values at which f is differentiable.**

87. $f(x) = |x + 3|$

88. $f(x) = \dfrac{2x}{x - 1}$

89. $f(x) = x^{2/5}$

90. $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \le 1 \\ x^2 - 2x, & x > 1 \end{cases}$

In Exercises 91–94, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

91. $f(x) = |x - 1|$

92. $f(x) = \sqrt{1 - x^2}$

93. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

94. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 95 and 96, determine whether the function is differentiable at $x = 2$.

95. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

96. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

97. ***Graphical Reasoning*** A line with slope m passes through the point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the function d in part (a). Based on the graph, is the function differentiable at every value of m? If not, where is it not differentiable?

98. ***Conjecture*** Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify a pattern between f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. The slope of the tangent line to the differentiable function f at the point $(2, f(2))$ is $\dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

100. If a function is continuous at a point, then it is differentiable at that point.

101. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

102. If a function is differentiable at a point, then it is continuous at that point.

103. Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

104. ***Writing*** Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

Section 2.2 Basic Differentiation Rules and Rates of Change

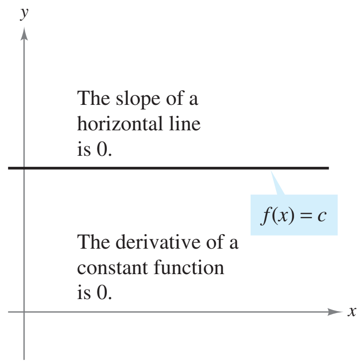
Video Video

Video Video

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Use derivatives to find rates of change.

The Constant Rule

In Section 2.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



The Constant Rule
Figure 2.14

NOTE In Figure 2.14, note that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

Proof Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0.\end{aligned}$$

EXAMPLE 1 Using the Constant Rule

Function	*Derivative*
a. $y = 7$	$\frac{dy}{dx} = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

Try It Exploration A

The editable graph feature below allows you to edit the graph of a function.

a. Editable Graph **b.** Editable Graph

c. Editable Graph **d.** Editable Graph

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each function. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$

d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Proof If n is a positive integer greater than 1, then the binomial expansion produces

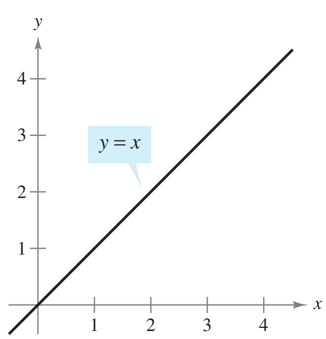
$$\begin{aligned}
\frac{d}{dx}[x^n] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right] \\
&= nx^{n-1} + 0 + \cdots + 0 \\
&= nx^{n-1}.
\end{aligned}$$

This proves the case for which n is a positive integer greater than 1. You will prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which n is a negative integer. In Exercise 75 in Section 2.5 you are asked to prove the case for which n is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of n.)

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{Power Rule when } n = 1$$

This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.



The slope of the line $y = x$ is 1.
Figure 2.15

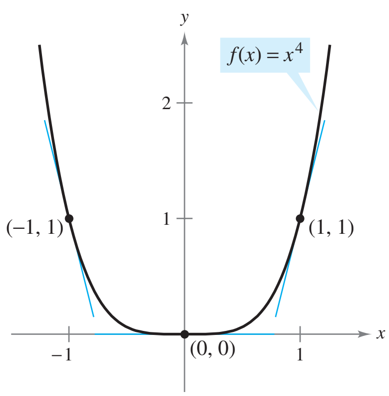
EXAMPLE 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$
c. $y = \dfrac{1}{x^2}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$

Try It | Exploration A

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given: $y = \dfrac{1}{x^2}$ ⇨ Rewrite: $y = x^{-2}$ ⇨ Differentiate: $\dfrac{dy}{dx} = (-2)x^{-3}$ ⇨ Simplify: $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$



Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.
Figure 2.16

Editable Graph

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

Solution The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

See Figure 2.16.

Try It | Exploration A | Open Exploration

EXAMPLE 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$(-2, f(-2)) = (-2, 4)$ Point on graph

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$m = f'(-2) = -4$ Slope of graph at $(-2, 4)$

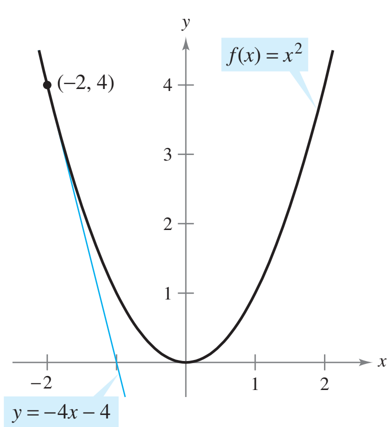
Now, using the point-slope form of the equation of a line, you can write

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 4 = -4[x - (-2)]$ Substitute for y_1, m, and x_1.

$y = -4x - 4.$ Simplify.

See Figure 2.17.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 2.17

Editable Graph

Try It | Exploration A | Exploration B | Open Exploration

The Constant Multiple Rule

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

Proof

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative}\\ &= \lim_{\Delta x \to 0} c\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]\\ &= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right] && \text{Apply Theorem 1.2.}\\ &= cf'(x)\end{aligned}$$

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{c}\right] &= \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right]\\ &= \left(\frac{1}{c}\right)\frac{d}{dx}[f(x)] = \left(\frac{1}{c}\right)f'(x)\end{aligned}$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	*Derivative*
a. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
b. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
c. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
d. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
e. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

Try It

Exploration A

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$D_x[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 Using Parentheses When Differentiating

Original Function	*Rewrite*	*Differentiate*	*Simplify*
a. $y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b. $y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c. $y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d. $y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

Try It **Exploration A**

The Sum and Difference Rules

THEOREM 2.5 The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Proof A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

EXAMPLE 7 Using the Sum and Difference Rules

Function	*Derivative*
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

Try It **Exploration A**

The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*.

MathArticle

Derivatives of Sine and Cosine Functions

In Section 1.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

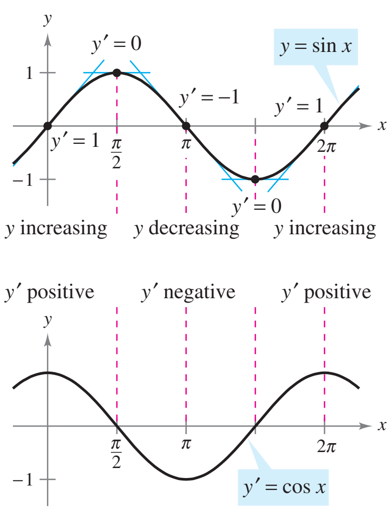
These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Proof

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$



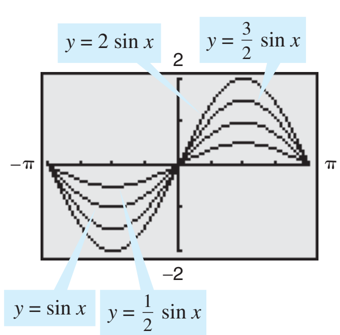
The derivative of the sine function is the cosine function.
Figure 2.18

Animation

This differentiation rule is shown graphically in Figure 2.18. Note that for each x, the *slope* of the sine curve is equal to the value of the cosine. The proof of the second rule is left as an exercise (see Exercise 116).

EXAMPLE 8 Derivatives Involving Sines and Cosines

	Function	*Derivative*
a.	$y = 2 \sin x$	$y' = 2 \cos x$
b.	$y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c.	$y = x + \cos x$	$y' = 1 - \sin x$



$\frac{d}{dx}[a \sin x] = a \cos x$
Figure 2.19

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

Try It | Exploration A | Open Exploration

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \qquad \text{Average velocity}$$

EXAMPLE 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

Solution

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

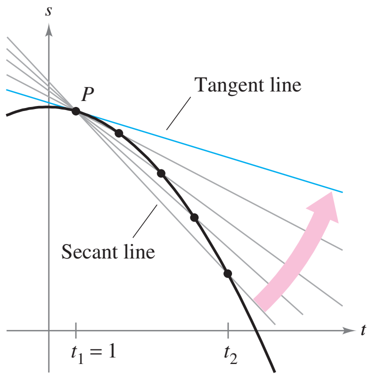
$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.

Try It | Exploration A | Exploration B



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 2.20

Animation

Suppose that in Example 9 you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

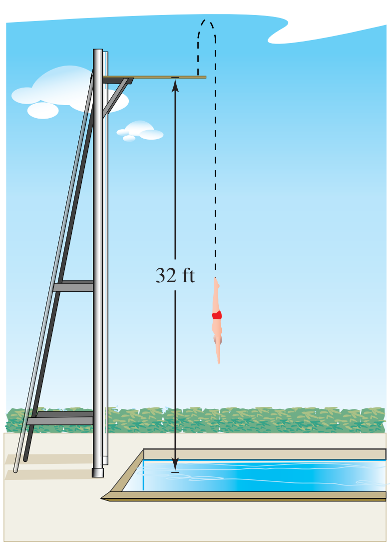
In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.

History



Velocity is positive when an object is rising, and is negative when an object is falling.
Figure 2.21

Animation

NOTE In Figure 2.21, note that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$ Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$ Factor.

$$t = -1 \text{ or } 2$$ Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. So, the velocity at time $t = 2$ is

$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

Try It Exploration A

Exercises for Section 2.2

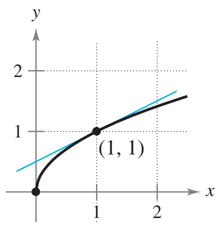
The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

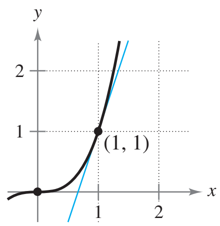
Click on M to print an enlarged copy of the graph.

In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point $(1, 1)$. Verify your answer analytically. To print an enlarged copy of the graph, select the MathGraph button.

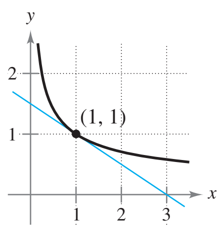
1. (a) $y = x^{1/2}$



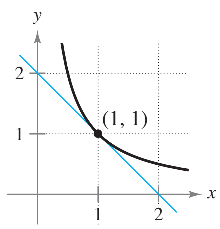
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



In Exercises 3–24, find the derivative of the function.

3. $y = 8$

4. $f(x) = -2$

5. $y = x^6$

6. $y = x^8$

7. $y = \dfrac{1}{x^7}$

8. $y = \dfrac{1}{x^8}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[4]{x}$

11. $f(x) = x + 1$

12. $g(x) = 3x - 1$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 + 2t - 3$

15. $g(x) = x^2 + 4x^3$

16. $y = 8 - x^3$

17. $s(t) = t^3 - 2t + 4$

18. $f(x) = 2x^3 - x^2 + 3x$

19. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

20. $g(t) = \pi\cos t$

21. $y = x^2 - \frac{1}{2}\cos x$

22. $y = 5 + \sin x$

23. $y = \dfrac{1}{x} - 3\sin x$

24. $y = \dfrac{5}{(2x)^3} + 2\cos x$

In Exercises 25–30, complete the table.

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
25.	$y = \dfrac{5}{2x^2}$			
26.	$y = \dfrac{2}{3x^2}$			
27.	$y = \dfrac{3}{(2x)^3}$			
28.	$y = \dfrac{\pi}{(3x)^2}$			
29.	$y = \dfrac{\sqrt{x}}{x}$			
30.	$y = \dfrac{4}{x^{-3}}$			

In Exercises 31–38, find the slope of the graph of the function a the given point. Use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
31.	$f(x) = \dfrac{3}{x^2}$	$(1, 3)$
32.	$f(t) = 3 - \dfrac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
33.	$f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34.	$y = 3x^3 - 6$	$(2, 18)$
35.	$y = (2x + 1)^2$	$(0, 1)$
36.	$f(x) = 3(5 - x)^2$	$(5, 0)$
37.	$f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38.	$g(t) = 2 + 3\cos t$	$(\pi, -1)$

In Exercises 39–52, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$

40. $f(x) = x^2 - 3x - 3x^{-2}$

41. $g(t) = t^2 - \dfrac{4}{t^3}$

42. $f(x) = x + \dfrac{1}{x^2}$

43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

44. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$

45. $y = x(x^2 + 1)$

46. $y = 3x(6x - 5x^2)$

47. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$

49. $h(s) = s^{4/5} - s^{2/3}$

50. $f(t) = t^{2/3} - t^{1/3} + 4$

51. $f(x) = 6\sqrt{x} + 5\cos x$

52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
53.	$y = x^4 - 3x^2 + 2$	$(1, 0)$
54.	$y = x^3 + x$	$(-1, -2)$
55.	$f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
56.	$y = (x^2 + 2x)(x + 1)$	$(1, 6)$

In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

57. $y = x^4 - 8x^2 + 2$

58. $y = x^3 + x$

59. $y = \dfrac{1}{x^2}$

60. $y = x^2 + 1$

61. $y = x + \sin x, \quad 0 \le x < 2\pi$

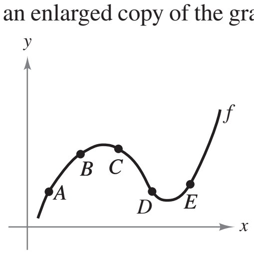
62. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

In Exercises 63–66, find k such that the line is tangent to the graph of the function.

	Function	*Line*
63.	$f(x) = x^2 - kx$	$y = 4x - 9$
64.	$f(x) = k - x^2$	$y = -4x + 7$
65.	$f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
66.	$f(x) = k\sqrt{x}$	$y = x + 4$

Writing About Concepts

67. Use the graph of f to answer each question. To print an enlarged copy of the graph, select the MathGraph button.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

68. Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

In Exercises 69 and 70, the relationship between f and g is given. Explain the relationship between f' and g'.

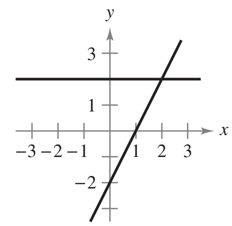
69. $g(x) = f(x) + 6$

70. $g(x) = -5f(x)$

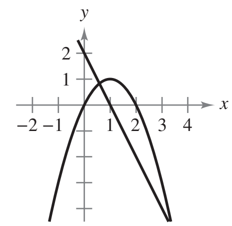
Writing About Concepts *(continued)*

In Exercises 71 and 72, the graphs of a function f and its derivative f' are shown on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, select the MathGraph button.

71.



72.



73. Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

74. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their point of intersection.

75. Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

76. Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

In Exercises 77 and 78, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

77. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

78. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

79. ***Linear Approximation*** Use a graphing utility, with a square window setting, to zoom in on the graph of

$$f(x) = 4 - \tfrac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.

80. ***Linear Approximation*** Use a graphing utility, with a square window setting, to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.

81. ***Linear Approximation*** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to graph f. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T on the same set of coordinate axes. Note that T is a good approximation of f when x is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

82. ***Linear Approximation*** Repeat Exercise 81 for the function $f(x) = x^3$ where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 81.

True or False? **In Exercises 83–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

84. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

85. If $y = \pi^2$, then $dy/dx = 2\pi$.

86. If $y = x/\pi$, then $dy/dx = 1/\pi$.

87. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

88. If $f(x) = 1/x^n$, then $f'(x) = 1/(nx^{n-1})$.

In Exercises 89–92, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

89. $f(t) = 2t + 7, \quad [1, 2]$

90. $f(t) = t^2 - 3, \quad [2, 2.1]$

91. $f(x) = \dfrac{-1}{x}, \quad [1, 2]$

92. $f(x) = \sin x, \quad \left[0, \dfrac{\pi}{6}\right]$

Vertical Motion **In Exercises 93 and 94, use the positio function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

93. A silver dollar is dropped from the top of a building that is 136 feet tall.

(a) Determine the position and velocity functions for the coin

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

94. A ball is thrown straight down from the top of a 220-foo building with an initial velocity of -22 feet per second. Wha is its velocity after 3 seconds? What is its velocity after fallin 108 feet?

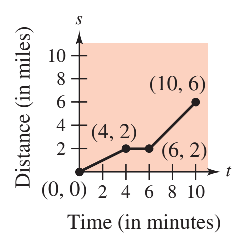
Vertical Motion **In Exercises 95 and 96, use the position func tion $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

95. A projectile is shot upward from the surface of Earth with a initial velocity of 120 meters per second. What is its velocit after 5 seconds? After 10 seconds?

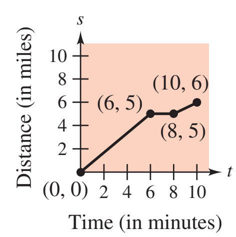
96. To estimate the height of a building, a stone is dropped from th top of the building into a pool of water at ground level. Ho high is the building if the splash is seen 6.8 seconds after th stone is dropped?

Think About It **In Exercises 97 and 98, the graph of a positio function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketc of the corresponding velocity function.**

97.

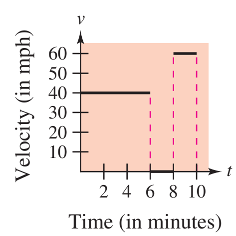


98.

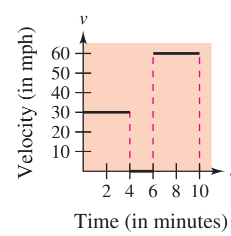


Think About It **In Exercises 99 and 100, the graph of a velocit function is shown. It represents the velocity in miles per hou during a 10-minute drive to work. Make a sketch of the corre sponding position function.**

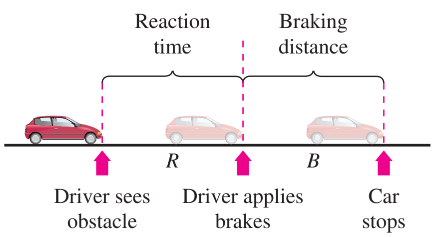
99.



100.



101. ***Modeling Data*** The stopping distance of an automobile, on dry, level pavement, traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



Speed, v	20	40	60	80	100
Reaction Time Distance, R	8.3	16.7	25.0	33.3	41.7
Braking Time Distance, B	2.3	9.0	20.2	35.8	55.9

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time distance.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking distance.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

102. ***Fuel Cost*** A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$1.55 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
dC/dx							

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon or the driver of a car that gets 35 miles per gallon? Explain.

103. ***Volume*** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$ centimeters.

104. ***Area*** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 4$ meters.

105. ***Velocity*** Verify that the average velocity over the tim interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneou velocity at $t = t_0$ for the position function

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

106. ***Inventory Management*** The annual inventory cost C for manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished Find the change in annual cost when Q is increased from 35 to 351, and compare this with the instantaneous rate of chang when $Q = 350$.

107. ***Writing*** The number of gallons N of regular unleade gasoline sold by a gasoline station at a price of p dollars pe gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(1.479)$.

(b) Is $f'(1.479)$ usually positive or negative? Explain.

108. ***Newton's Law of Cooling*** This law states that the rate o change of the temperature of an object is proportional to th difference between the object's temperature T and th temperature T_a of the surrounding medium. Write an equatio for this law.

109. Find an equation of the parabola $y = ax^2 + bx + c$ that passe through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

110. Let (a, b) be an arbitrary point on the graph of $y = 1/x$ $x > 0$. Prove that the area of the triangle formed by th tangent line through (a, b) and the coordinate axes is 2.

111. Find the tangent line(s) to the curve $y = x^3 - 9x$ through th point $(1, -9)$.

112. Find the equation(s) of the tangent line(s) to the parabol $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

In Exercises 113 and 114, find a and b such that f is differentiable everywhere.

113. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

114. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

115. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x$ differentiable?

116. Prove that $\dfrac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*.

MathArticle

Section 2.3 Product and Quotient Rules and Higher-Order Derivatives

- **Find the derivative of a function using the Product Rule.**
- **Find the derivative of a function using the Quotient Rule.**
- **Find the derivative of a trigonometric function.**
- **Find a higher-order derivative of a function.**

The Product Rule

In Section 2.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

NOTE A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products involving three or more factors.

Video

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x(-\sin x) \\
&= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

The Product Rule

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. (*Source: The History of Mathematics by David M. Burton*)

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$h'(x) = \underbrace{(3x - 2x^2)}_{\text{First}} \underbrace{\frac{d}{dx}[5 + 4x]}_{\text{Derivative of second}} + \underbrace{(5 + 4x)}_{\text{Second}} \underbrace{\frac{d}{dx}[3x - 2x^2]}_{\text{Derivative of first}} \qquad \text{Apply Product Rule.}$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= -24x^2 + 4x + 15$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$D_x[(3x - 2x^2)(5 + 4x)] = D_x[-8x^3 + 2x^2 + 15x]$$
$$= -24x^2 + 4x + 15.$$

Try It | Exploration A

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\frac{d}{dx}[3x^2 \sin x] = 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] \qquad \text{Apply Product Rule.}$$

$$= 3x^2 \cos x + (\sin x)(6x)$$

$$= 3x^2 \cos x + 6x \sin x$$

$$= 3x(x \cos x + 2 \sin x)$$

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The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\frac{dy}{dx} = \underbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}_{\text{Product Rule}} - \underbrace{2\frac{d}{dx}[\sin x]}_{\text{Constant Multiple Rule}}$$

$$= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$$

$$= -2x \sin x$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

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The Quotient Rule

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0$$

Video

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

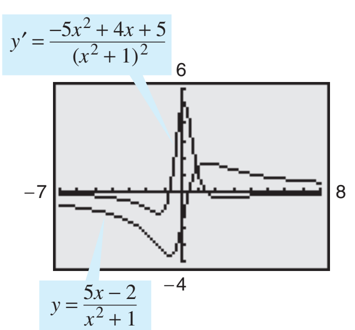
$$= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Note that $\lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative

Figure 2.22

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x-2}{x^2+1}$.

Solution

$$\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] = \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} \qquad \text{Apply Quotient Rule.}$$

$$= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}$$

$$= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2}$$

$$= \frac{-5x^2+4x+5}{(x^2+1)^2}$$

Try It | Exploration A | Exploration B

The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$f(x) = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by } x.$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

$$f'(x) = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Quotient Rule}$$

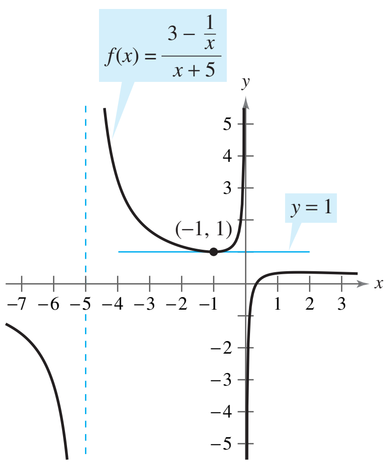
$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 2.23

Try It | Exploration A

The editable graph feature below allows you to edit the graph of a function.

Editable Graph

Not every quotient needs to be differentiated by the Quotient Rule. For example, each quotient in the next example can be considered as the product of a constant times a function of x. In such cases it is more convenient to use the Constant Multiple Rule.

EXAMPLE 6 Using the Constant Multiple Rule

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

Try It | Exploration A

In Section 2.2, the Power Rule was proved only for the case where the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right]\\ &= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule}\\ &= \frac{0 - kx^{k-1}}{x^{2k}}\\ &= -kx^{-k-1}\\ &= nx^{n-1}. && n = -k\end{aligned}$$

So, the Power Rule

$$D_x[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 75 in Section 2.5, you are asked to prove the case for which n is any rational number.

Try It

Exploration A

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Video

Proof Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 89).

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

EXAMPLE 8 Differentiating Trigonometric Functions

Function	*Derivative*
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$
	$= (\sec x)(1 + x \tan x)$

Try It Exploration A Open Exploration

EXAMPLE 9 Different Forms of a Derivative

Differentiate both forms of $y = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x$.

Solution

First form: $y = \dfrac{1 - \cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x}$$

Second form: $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= \csc^2 x - \csc x \cot x.$$

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The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ *After Differentiating*	$f'(x)$ *After Simplifying*
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 9	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$s(t) \quad \text{Position function}$$
$$v(t) = s'(t) \quad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \quad \text{Acceleration function}$$

NOTE: The second derivative of f is the derivative of the first derivative of f.

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
⋮					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \quad \text{Position function}$$
$$s'(t) = -1.62t \quad \text{Velocity function}$$
$$s''(t) = -1.62 \quad \text{Acceleration function}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.05.$$

THE MOON

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.03.$$

Video

Try It

Exploration A

Exercises for Section 2.3

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

2. $f(x) = (6x + 5)(x^3 - 2)$

3. $h(t) = \sqrt[3]{t}(t^2 + 4)$

4. $g(s) = \sqrt{s}(4 - s^2)$

5. $f(x) = x^3 \cos x$

6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$

8. $g(t) = \dfrac{t^2 + 2}{2t - 7}$

9. $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1}$

10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$

11. $g(x) = \dfrac{\sin x}{x^2}$

12. $f(t) = \dfrac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

	Function	*Value of c*
13.	$f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$	$c = 0$
14.	$f(x) = (x^2 - 2x + 1)(x^3 - 1)$	$c = 1$
15.	$f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16.	$f(x) = \dfrac{x + 1}{x - 1}$	$c = 2$
17.	$f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18.	$f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

In Exercises 19–24, complete the table without using the Quotient Rule.

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 2x}{3}$			
20.	$y = \dfrac{5x^2 - 3}{4}$			
21.	$y = \dfrac{7}{3x^3}$			
22.	$y = \dfrac{4}{5x^2}$			
23.	$y = \dfrac{4x^{3/2}}{x}$			
24.	$y = \dfrac{3x^2 - 5}{7}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

26. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$

28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$

29. $f(x) = \dfrac{2x + 5}{\sqrt{x}}$

30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$

31. $h(s) = (s^3 - 2)^2$

32. $h(x) = (x^2 - 1)^2$

33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$

34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$

35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant

38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$

40. $f(\theta) = (\theta + 1) \cos \theta$

41. $f(t) = \dfrac{\cos t}{t}$

42. $f(x) = \dfrac{\sin x}{x}$

43. $f(x) = -x + \tan x$

44. $y = x + \cot x$

45. $g(t) = \sqrt[4]{t} + 8 \sec t$

46. $h(s) = \dfrac{1}{s} - 10 \csc s$

47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$

48. $y = \dfrac{\sec x}{x}$

49. $y = -\csc x - \sin x$

50. $y = x \sin x + \cos x$

51. $f(x) = x^2 \tan x$

52. $f(x) = \sin x \cos x$

53. $y = 2x \sin x + x^2 \cos x$

54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

In Exercises 55–58, use a computer algebra system to differentiate the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$

56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$

58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

	Function	*Point*
59.	$y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60.	$f(x) = \tan x \cot x$	$(1, 1)$
61.	$h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62.	$f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

63. $f(x) = (x^3 - 3x + 1)(x + 2)$, $(1, -3)$

64. $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

65. $f(x) = \dfrac{x}{x - 1}$, $(2, 2)$

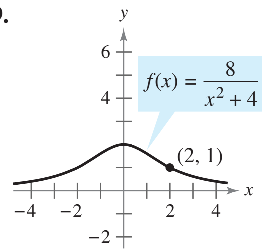
66. $f(x) = \dfrac{(x - 1)}{(x + 1)}$, $\left(2, \dfrac{1}{3}\right)$

67. $f(x) = \tan x$, $\left(\dfrac{\pi}{4}, 1\right)$

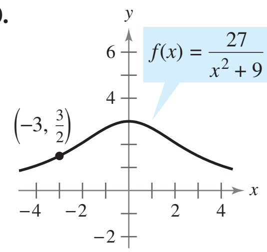
68. $f(x) = \sec x$, $\left(\dfrac{\pi}{3}, 2\right)$

Famous Curves **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

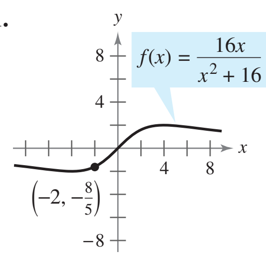
69.



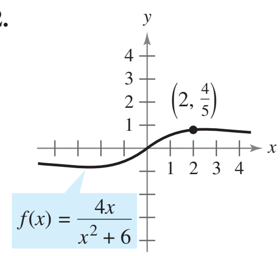
70.



71.



72.



In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.

73. $f(x) = \dfrac{x^2}{x - 1}$

74. $f(x) = \dfrac{x^2}{x^2 + 1}$

75. $f(x) = \dfrac{4x - 2}{x^2}$

76. $f(x) = \dfrac{x - 4}{x^2 - 7}$

77. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = \dfrac{x + 1}{x - 1}$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

78. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = \dfrac{x}{x - 1}$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explaii the relationship between f and g.

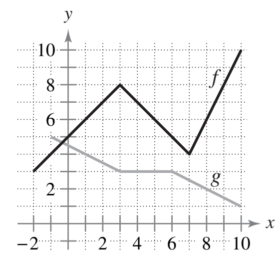
79. $f(x) = \dfrac{3x}{x + 2}$, $g(x) = \dfrac{5x + 4}{x + 2}$

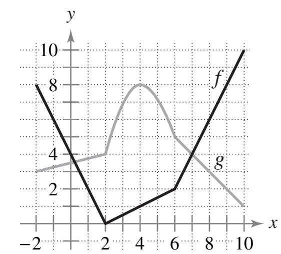
80. $f(x) = \dfrac{\sin x - 3x}{x}$, $g(x) = \dfrac{\sin x + 2x}{x}$

In Exercises 81 and 82, use the graphs of f and g. Le $p(x) = f(x)g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$.

81. (a) Find $p'(1)$.
(b) Find $q'(4)$.

82. (a) Find $p'(4)$.
(b) Find $q'(7)$.





83. ***Area*** The length of a rectangle is given by $2t + 1$ and it: height is $\sqrt{t}$, where t is time in seconds and the dimensions ar in centimeters. Find the rate of change of the area with respec to time.

84. ***Volume*** The radius of a right circular cylinder is given b $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and th dimensions are in inches. Find the rate of change of the volum with respect to time.

85. ***Inventory Replenishment*** The ordering and transportatio cost C for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the orde size in hundreds. Find the rate of change of C with respect to when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do thes rates of change imply about increasing order size?

86. ***Boyle's Law*** This law states that if the temperature of a ga remains constant, its pressure is inversely proportional to it volume. Use the derivative to show that the rate of change of th pressure is inversely proportional to the square of the volume.

87. ***Population Growth*** A population of 500 bacteria is introduce into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the popu lation is growing when $t = 2$.

88. ***Gravitational Force*** Newton's Law of Universal Gravitation states that the force F between two masses, m_1 and m_2, is

$$F = \frac{Gm_1m_2}{d^2}$$

where G is a constant and d is the distance between the masses. Find an equation that gives an instantaneous rate of change of F with respect to d. (Assume m_1 and m_2 represent moving points.)

89. Prove the following differentiation rules.

(a) $\frac{d}{dx}[\sec x] = \sec x \tan x$ (b) $\frac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\frac{d}{dx}[\cot x] = -\csc^2 x$

90. ***Rate of Change*** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

91. ***Modeling Data*** The table shows the numbers n (in thousands) of motor homes sold in the United States and the retail values v (in billions of dollars) of these motor homes for the years 1996 through 2001. The year is represented by t, with $t = 6$ corresponding to 1996. (*Source: Recreation Vehicle Industry Association*)

Year, t	6	7	8	9	10	11
n	247.5	254.5	292.7	321.2	300.1	256.8
v	6.3	6.9	8.4	10.4	9.5	8.6

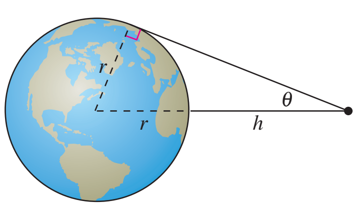
(a) Use a graphing utility to find cubic models for the number of motor homes sold $n(t)$ and the total retail value $v(t)$ of the motor homes.

(b) Graph each model found in part (a).

(c) Find $A = v(t)/n(t)$, then graph A. What does this function represent?

(d) Interpret $A'(t)$ in the context of these data.

92. ***Satellites*** When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface and let r represent Earth's radius.



(a) Show that $h = r(\csc\theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30°$. (Assume $r = 3960$ miles.)

In Exercises 93–98, find the second derivative of the function.

93. $f(x) = 4x^{3/2}$ **94.** $f(x) = x + 32x^{-2}$

95. $f(x) = \frac{x}{x-1}$ **96.** $f(x) = \frac{x^2 + 2x - 1}{x}$

97. $f(x) = 3 \sin x$ **98.** $f(x) = \sec x$

In Exercises 99–102, find the given higher-order derivative.

99. $f'(x) = x^2$, $f''(x)$ **100.** $f''(x) = 2 - \frac{2}{x}$, $f'''(x)$

101. $f'''(x) = 2\sqrt{x}$, $f^{(4)}(x)$ **102.** $f^{(4)}(x) = 2x + 1$, $f^{(6)}(x$

Writing About Concepts

103. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$.

104. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x.

In Exercises 105–108, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

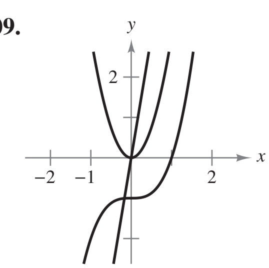
$h(2) = -1$ and $h'(2) = 4$

105. $f(x) = 2g(x) + h(x)$ **106.** $f(x) = 4 - h(x)$

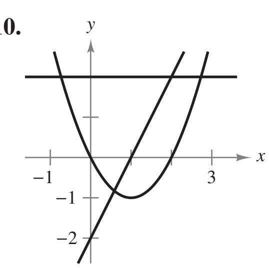
107. $f(x) = \frac{g(x)}{h(x)}$ **108.** $f(x) = g(x)h(x)$

In Exercises 109 and 110, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, select the MathGraph button.

109.

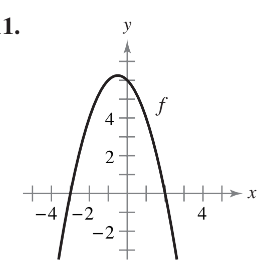


110.

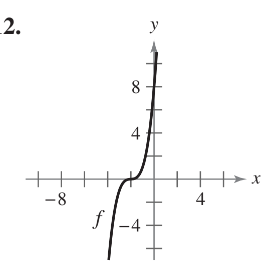


In Exercises 111–114, the graph of f is shown. Sketch th graphs of f' and f''. To print an enlarged copy of the graph select the MathGraph button.

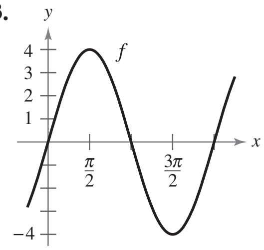
111.



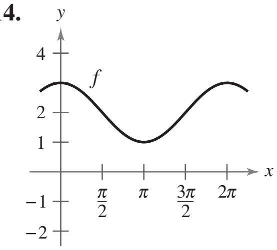
112.



113.



114.



115. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

116. ***Acceleration*** An automobile's velocity starting from rest is

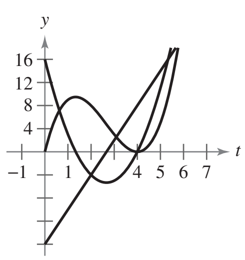
$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

117. ***Stopping Distance*** A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

118. ***Particle Motion*** The figure shows the graphs of the position, velocity, and acceleration functions of a particle.



(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, select the MathGraph button.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

Finding a Pattern **In Exercises 119 and 120, develop a general rule for $f^{(n)}(x)$ given $f(x)$.**

119. $f(x) = x^n$ **120.** $f(x) = \dfrac{1}{x}$

121. ***Finding a Pattern*** Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results in part (a) to write a general rule for $f^{(n)}(x)$.

122. ***Finding a Pattern*** Develop a general rule for $[xf(x)]^{(n)}$ where f is a differentiable function of x.

In Exercises 123 and 124, find the derivatives of the function f for $n = 1, 2, 3$, and 4. Use the results to write a general rule for $f'(x)$ in terms of n.

123. $f(x) = x^n \sin x$ **124.** $f(x) = \dfrac{\cos x}{x^n}$

Differential Equations **In Exercises 125–128, verify that the function satisfies the differential equation.**

	Function	*Differential Equation*
125.	$y = \dfrac{1}{x}, x > 0$	$x^3 y'' + 2x^2 y' = 0$
126.	$y = 2x^3 - 6x + 10$	$-y''' - xy'' - 2y' = -24x^2$
127.	$y = 2 \sin x + 3$	$y'' + y = 3$
128.	$y = 3 \cos x + \sin x$	$y'' + y = 0$

True or False? **In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

130. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

131. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

132. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

133. The second derivative represents the rate of change of the first derivative.

134. If the velocity of an object is constant, then its acceleration is zero.

135. Find a second-degree polynomial $f(x) = ax^2 + bx + c$ such that its graph has a tangent line with slope 10 at the point $(2, 7)$ and an x-intercept at $(1, 0)$.

136. Consider the third-degree polynomial

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

Determine conditions for a, b, c, and d if the graph of f has (a) no horizontal tangents, (b) exactly one horizontal tangent, and (c) exactly two horizontal tangents. Give an example for each case.

137. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

138. ***Think About It*** Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$

(b) $fg'' + f''g = (fg)''$

Section 2.4 The Chain Rule

- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a trigonometric function using the Chain Rule.**

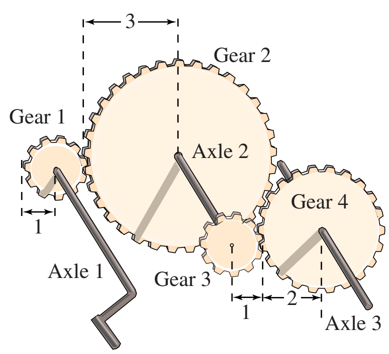
The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best done with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

Video



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 2.24

Animation

EXAMPLE 1 The Derivative of a Composite Function

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\frac{dy}{dx} = \begin{pmatrix}\text{Rate of change of first axle} \\ \text{with respect to second axle}\end{pmatrix} \cdot \begin{pmatrix}\text{Rate of change of second axle} \\ \text{with respect to third axle}\end{pmatrix}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

$$= \begin{pmatrix}\text{Rate of change of first axle} \\ \text{with respect to third axle}\end{pmatrix}.$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

Try It   Exploration A

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x + 1}$

b. $(x + 2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated below.

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Proof Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\
&= \left[\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right] \\
&= f'(g(c))g'(c)
\end{aligned}$$

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

EXAMPLE 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

Try It   Exploration A

EXAMPLE 3 Using the Chain Rule

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$

Try It   Exploration A   Exploration B

The editable graph feature below allows you to edit the graph of a function and its derivative.

Editable Graph

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

Video

Proof Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = n u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

Video

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

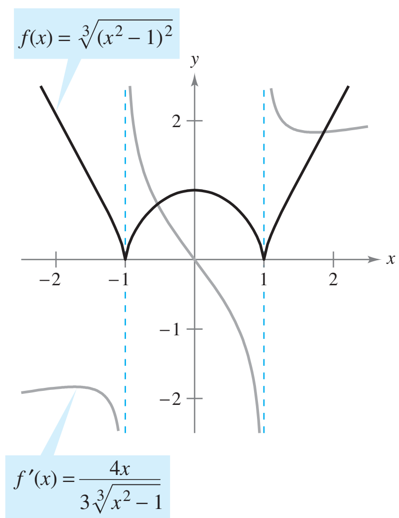
$$f'(x) = \overbrace{3}^{n}\overbrace{(3x - 2x^2)^2}^{u^{n-1}} \overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'} \qquad \text{Apply General Power Rule.}$$

$$= 3(3x - 2x^2)^2(3 - 4x). \qquad \text{Differentiate } 3x - 2x^2.$$

Try It | Exploration A

The editable graph feature below allows you to edit the graph of a function.

Editable Graph



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.25

Editable Graph

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$f'(x) = \overbrace{\frac{2}{3}}^{n}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} \qquad \text{Apply General Power Rule.}$$

$$= \frac{4x}{3\sqrt[3]{x^2 - 1}}. \qquad \text{Write in radical form.}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 2.25.

Try It | Exploration A

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule produces

$$g'(t) = \underbrace{(-7)}_{\text{Constant Multiple Rule}}\overbrace{(-2)}^{n}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} \qquad \text{Apply General Power Rule.}$$

$$= 28(2t - 3)^{-3} \qquad \text{Simplify.}$$

$$= \frac{28}{(2t - 3)^3}. \qquad \text{Write with positive exponent.}$$

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Try It | Exploration A | Exploration B

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

$$
\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Original function}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}
$$

Try It Exploration A

EXAMPLE 8 Simplifying the Derivative of a Quotient

TECHNOLOGY Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given on this page.

$$
\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4)-(2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}
$$

Try It Exploration A

EXAMPLE 9 Simplifying the Derivative of a Power

$$
\begin{aligned}
y &= \left(\frac{3x-1}{x^2+3}\right)^2 && \text{Original function}\\
y' &= \underbrace{2}_{n}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}
$$

Try It Exploration A Open Exploration

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as shown.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

Technology

EXAMPLE 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \underbrace{2x}_{u}$ $\qquad y' = \underbrace{\cos 2x}_{\cos u}\underbrace{\frac{d}{dx}[2x]}_{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x - 1)$ $\qquad y' = -\sin(x - 1)$

c. $y = \tan 3x$ $\qquad y' = 3\sec^2 3x$

Try It Exploration A

Be sure that you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), sin $2x$ is written to mean $\sin(2x)$.

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$

Try It Exploration A

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$ — Original function

$= (\sin 4t)^3$ — Rewrite.

$f'(t) = 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t]$ — Apply Chain Rule once.

$= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t]$ — Apply Chain Rule a second time.

$= 3(\sin 4t)^2(\cos 4t)(4)$

$= 12\sin^2 4t \cos 4t$ — Simplify.

Try It Exploration A

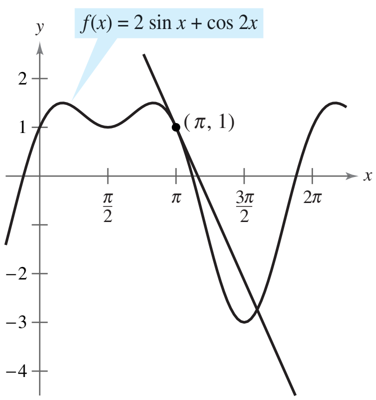


Figure 2.26

STUDY TIP To become skilled at differentiation, you should memorize each rule. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

EXAMPLE 13 Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of

$$f(x) = 2\sin x + \cos 2x$$

at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Solution Begin by finding $f'(x)$.

$$f(x) = 2\sin x + \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2\cos x + (-\sin 2x)(2) \qquad \text{Apply Chain Rule to } \cos 2x.$$

$$= 2\cos x - 2\sin 2x \qquad \text{Simplify.}$$

To find the equation of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$f'(\pi) = 2\cos\pi - 2\sin 2\pi \qquad \text{Substitute.}$$

$$= -2 \qquad \text{Slope of graph at } (\pi, 1)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 1 = -2(x - \pi) \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$

$$y = 1 - 2x + 2\pi. \qquad \text{Equation of tangent line at } (\pi, 1)$$

You can then determine that $f'(x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. So, f has a horizontal tangent at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Try It | Exploration A

This section concludes with a summary of the differentiation rules studied so far.

Summary of Differentiation Rules

General Differentiation Rules Let f, g, and u be differentiable functions of x.

Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$

Sum or Difference Rule: $\frac{d}{dx}[f \pm g] = f' \pm g'$

Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$

Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

Derivatives of Algebraic Functions

Constant Rule: $\frac{d}{dx}[c] = 0$

(Simple) Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$

Chain Rule

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\,u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}u'$

Exercises for Section 2.4

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on **S** to view the complete solution of the exercise.

Click on **M** to print an enlarged copy of the graph.

In Exercises 1–6, complete the table.

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (6x - 5)^4$		
2.	$y = \dfrac{1}{\sqrt{x+1}}$		
3.	$y = \sqrt{x^2 - 1}$		
4.	$y = 3\tan(\pi x^2)$		
5.	$y = \csc^3 x$		
6.	$y = \cos\dfrac{3x}{2}$		

In Exercises 7–32, find the derivative of the function.

7. $y = (2x - 7)^3$

8. $y = 3(4 - x^2)^5$

9. $g(x) = 3(4 - 9x)^4$

10. $f(t) = (9t + 2)^{2/3}$

11. $f(t) = \sqrt{1 - t}$

12. $g(x) = \sqrt{5 - 3x}$

13. $y = \sqrt[3]{9x^2 + 4}$

14. $g(x) = \sqrt{x^2 - 2x + 1}$

15. $y = 2\sqrt[4]{4 - x^2}$

16. $f(x) = -3\sqrt[4]{2 - 9x}$

17. $y = \dfrac{1}{x - 2}$

18. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

19. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

20. $y = -\dfrac{5}{(t + 3)^3}$

21. $y = \dfrac{1}{\sqrt{x + 2}}$

22. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

23. $f(x) = x^2(x - 2)^4$

24. $f(x) = x(3x - 9)^3$

25. $y = x\sqrt{1 - x^2}$

26. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

31. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

In Exercises 33–38, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

33. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

34. $y = \sqrt{\dfrac{2x}{x + 1}}$

35. $y = \sqrt{\dfrac{x + 1}{x}}$

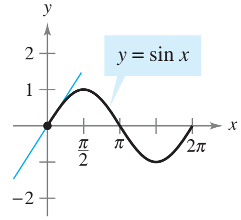
36. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

37. $y = \dfrac{\cos \pi x + 1}{x}$

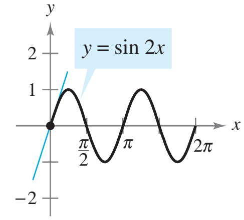
38. $y = x^2 \tan\dfrac{1}{x}$

In Exercises 39 and 40, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

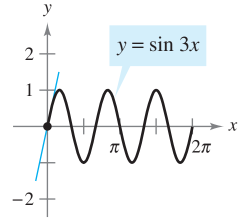
39. (a)



(b)



40. (a)



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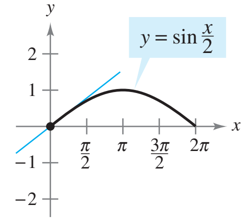


In Exercises 41–58, find the derivative of the function.

41. $y = \cos 3x$

42. $y = \sin \pi x$

43. $g(x) = 3\tan 4x$

44. $h(x) = \sec x^2$

45. $y = \sin(\pi x)^2$

46. $y = \cos(1 - 2x)^2$

47. $h(x) = \sin 2x \cos 2x$

48. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

49. $f(x) = \dfrac{\cot x}{\sin x}$

50. $g(v) = \dfrac{\cos v}{\csc v}$

51. $y = 4\sec^2 x$

52. $g(t) = 5\cos^2 \pi t$

53. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

54. $h(t) = 2\cot^2(\pi t + 2)$

55. $f(t) = 3\sec^2(\pi t - 1)$

56. $y = 3x - 5\cos(\pi x)^2$

57. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

58. $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

In Exercises 59–66, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

	Function	*Point*
59.	$s(t) = \sqrt{t^2 + 2t + 8}$	$(2, 4)$
60.	$y = \sqrt[5]{3x^3 + 4x}$	$(2, 2)$
61.	$f(x) = \dfrac{3}{x^3 - 4}$	$\left(-1, -\dfrac{3}{5}\right)$
62.	$f(x) = \dfrac{1}{(x^2 - 3x)^2}$	$\left(4, \dfrac{1}{16}\right)$
63.	$f(t) = \dfrac{3t + 2}{t - 1}$	$(0, -2)$
64.	$f(x) = \dfrac{x + 1}{2x - 3}$	$(2, 3)$
65.	$y = 37 - \sec^3(2x)$	$(0, 36)$
66.	$y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 67–74, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

Function	*Point*
67. $f(x) = \sqrt{3x^2 - 2}$	$(3, 5)$
68. $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$	$(2, 2)$
69. $y = (2x^3 + 1)^2$	$(-1, 1)$
70. $f(x) = (9 - x^2)^{2/3}$	$(1, 4)$
71. $f(x) = \sin 2x$	$(\pi, 0)$
72. $y = \cos 3x$	$\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
73. $f(x) = \tan^2 x$	$\left(\frac{\pi}{4}, 1\right)$
74. $y = 2\tan^3 x$	$\left(\frac{\pi}{4}, 2\right)$

In Exercises 75–78, (a) use a graphing utility to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the given point, and (c) use the utility to graph the function and its tangent line in the same viewing window.

75. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}, \quad \left(\frac{1}{2}, \frac{3}{2}\right)$

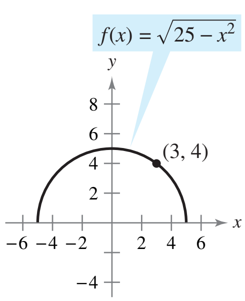
76. $f(x) = \sqrt{x}(2 - x)^2, \quad (4, 8)$

77. $s(t) = \dfrac{(4 - 2t)\sqrt{1 + t}}{3}, \quad \left(0, \frac{4}{3}\right)$

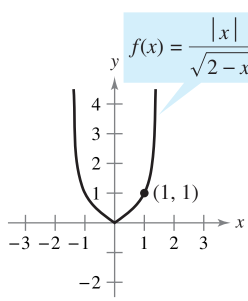
78. $y = (t^2 - 9)\sqrt{t + 2}, \quad (2, -10)$

Famous Curves **In Exercises 79 and 80, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

79. Top half of circle



80. Bullet-nose curve



81. ***Horizontal Tangent Line*** Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of $f(x) = 2\cos x + \sin 2x$ has a horizontal tangent.

82. ***Horizontal Tangent Line*** Determine the point(s) at which the graph of $f(x) = \dfrac{x}{\sqrt{2x - 1}}$ has a horizontal tangent.

In Exercises 83–86, find the second derivative of the function.

83. $f(x) = 2(x^2 - 1)^3$

84. $f(x) = \dfrac{1}{x - 2}$

85. $f(x) = \sin x^2$

86. $f(x) = \sec^2 \pi x$

In Exercises 87–90, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.

87. $h(x) = \frac{1}{9}(3x + 1)^3, \quad \left(1, \frac{64}{9}\right)$

88. $f(x) = \dfrac{1}{\sqrt{x + 4}}, \quad \left(0, \frac{1}{2}\right)$

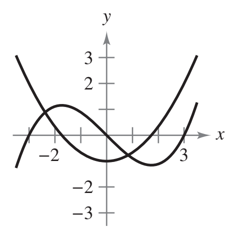
89. $f(x) = \cos(x^2), \quad (0, 1)$

90. $g(t) = \tan 2t, \quad \left(\frac{\pi}{6}, \sqrt{3}\right)$

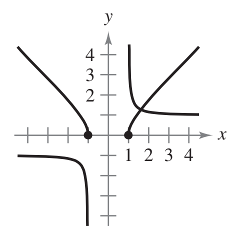
Writing About Concepts

In Exercises 91–94, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, select the MathGraph button.

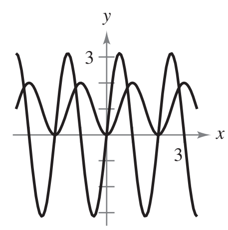
91.



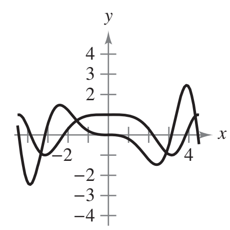
92.



93.



94.



In Exercises 95 and 96, the relationship between f and g is given. Explain the relationship between f' and g'.

95. $g(x) = f(3x)$

96. $g(x) = f(x^2)$

97. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

98. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

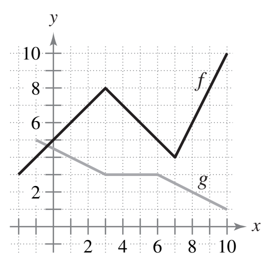
(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

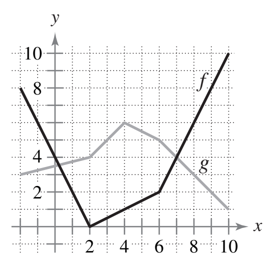
x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

In Exercises 99 and 100, the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

99. (a) Find $h'(1)$.
(b) Find $s'(5)$.



100. (a) Find $h'(3)$.
(b) Find $s'(9)$.



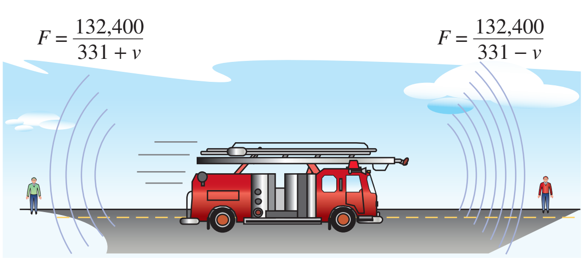
101. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



102. ***Harmonic Motion*** The displacement from equilibrium of a object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds Determine the position and velocity of the object whe $t = \pi/8$.

103. ***Pendulum*** A 15-centimeter pendulum moves according t the equation $\theta = 0.2\cos 8t$, where θ is the angular displace ment from the vertical in radians and t is the time in seconds Determine the maximum angular displacement and the rate o change of θ when $t = 3$ seconds.

104. ***Wave Motion*** A buoy oscillates in simple harmonic motio $y = A\cos \omega t$ as waves move past it. The buoy moves a tota of 3.5 feet (vertically) from its low point to its high point. I returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if i is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

105. ***Circulatory System*** The speed S of blood that is r centime ters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and is measured in centimeters per second. Suppose a drug i administered and the artery begins to dilate at a rate of dR/dt At a constant distance r, find the rate at which S changes wit respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, an $dR/dt = 10^{-5}$.

106. ***Modeling Data*** The normal daily maximum temperature T (in degrees Fahrenheit) for Denver, Colorado, are shown i the table. *(Source: National Oceanic and Atmospheri Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	43.2	47.2	53.7	60.9	70.5	82.1

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	88.0	86.0	77.4	66.0	51.5	44.1

(a) Use a graphing utility to plot the data and find a model fo the data of the form

$$T(t) = a + b\sin(\pi t/6 - c)$$

where T is the temperature and t is the time in months with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well doe the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative

(d) Based on the graph of the derivative, during what time does the temperature change most rapidly? Most slowly Do your answers agree with your observations of th temperature changes? Explain.

107. ***Modeling Data*** The cost of producing x units of a product is $C = 60x + 1350$. For one week management determined the number of units produced at the end of t hours during an eight-hour shift. The average values of x for the week are shown in the table.

t	0	1	2	3	4	5	6	7	8
x	0	16	60	130	205	271	336	384	392

(a) Use a graphing utility to fit a cubic model to the data.

(b) Use the Chain Rule to find dC/dt.

(c) Explain why the cost function is not increasing at a constant rate during the 8-hour shift.

108. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results in part (a) to write general rules for the even- and odd-order derivatives

$f^{(2k)}(x)$ and $f^{(2k-1)}(x)$.

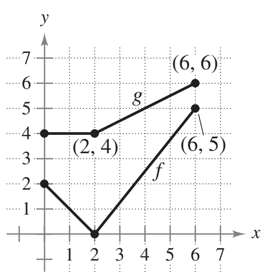
[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

109. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

110. ***Think About It*** Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$ where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



111. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$f'(x) = g'(x)$.

112. (a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

113. Let u be a differentiable function of x. Use the fact tha $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 114–117, use the result of Exercise 113 to find th derivative of the function.

114. $g(x) = |2x - 3|$

115. $f(x) = |x^2 - 4|$

116. $h(x) = |x| \cos x$

117. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quad ratic approximations of a function f at $x = a$ are**

$P_1(x) = f'(a)(x - a) + f(a)$ **and**

$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$.

In Exercises 118 and 119, (a) find the specified linear an quadratic approximations of f, (b) use a graphing utility t graph f and the approximations, (c) determine whether P_1 o P_2 is the better approximation, and (d) state how the accurac changes as you move farther from $x = a$.

118. $f(x) = \tan \dfrac{\pi x}{4}$, $a = 1$

119. $f(x) = \sec 2x$, $a = \dfrac{\pi}{6}$

True or False? **In Exercises 120–122, determine whether th statement is true or false. If it is false, explain why or give a example that shows it is false.**

120. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

121. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

122. If y is a differentiable function of u, u is a differentiabl function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

Putnam Exam Challenge

123. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdot \cdot \cdot + a_n \sin nx$, wher $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positiv integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove tha $|a_1 + 2a_2 + \cdot \cdot \cdot + na_n| \leq 1$.

124. Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k - 1}$ has the form

$$\frac{P_n(x)}{(x^k - 1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Section 2.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**

EXPLORATION

Graphing an Implicit Equation
How could you use a graphing utility to sketch the graph of the equation

$$x^2 - 2y^3 + 4y = 2?$$

Here are two possible approaches.

a. Solve the equation for x. Switch the roles of x and y and graph the two resulting equations. The combined graphs will show a 90° rotation of the graph of the original equation.

b. Set the graphing utility to *parametric* mode and graph the equations

$$x = -\sqrt{2t^3 - 4t + 2}$$
$$y = t$$

and

$$x = \sqrt{2t^3 - 4t + 2}$$
$$y = t.$$

From either of these two approaches, can you decide whether the graph has a tangent line at the point $(0, 1)$? Explain your reasoning.

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation

$$x^2 - 2y^3 + 4y = 2$$

where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to* x. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

Video

EXAMPLE 1 Differentiating with Respect to x

a. $\frac{d}{dx}[x^3] = 3x^2$ — Variables agree: use Simple Power Rule.

Variables agree

b. $\frac{d}{dx}[\underbrace{y^3}_{u^n}] = \underbrace{3y^2}_{nu^{n-1}}\underbrace{\frac{dy}{dx}}_{u'}$ — Variables disagree: use Chain Rule.

Variables disagree

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ — Chain Rule: $\frac{d}{dx}[3y] = 3y'$

d. $\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x]$ — Product Rule

$= x\left(2y\frac{dy}{dx}\right) + y^2(1)$ — Chain Rule

$= 2xy\frac{dy}{dx} + y^2$ — Simplify.

Try It Exploration A

Implicit Differentiation

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation and move all other terms to the right side of the equation.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

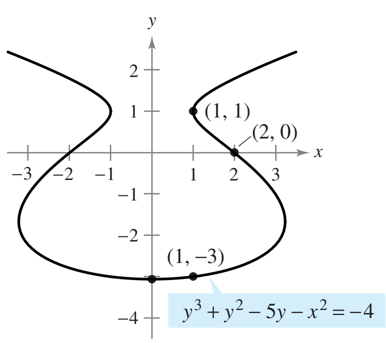
$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Try It | Exploration A | Video | Video

NOTE In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.



Point on Graph	*Slope of Graph*
$(2, 0)$	$-\frac{4}{5}$
$(1, -3)$	$\frac{1}{8}$
$x = 0$	0
$(1, 1)$	Undefined

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

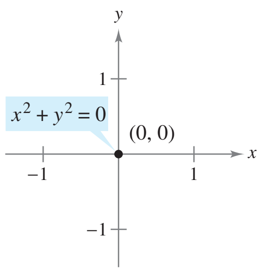
has the derivative

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Figure 2.27

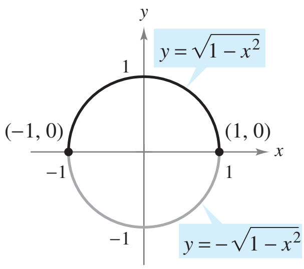
To see how you can use an *implicit derivative*, consider the graph shown in Figure 2.27. From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.

TECHNOLOGY With most graphing utilities, it is easy to graph an equation that explicitly represents y as a function of x. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 2.27?



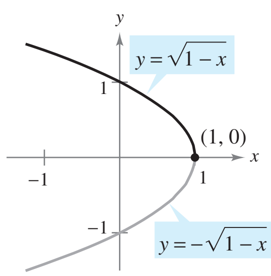
(a)

Editable Graph



(b)

Editable Graph



(c)

Editable Graph

Some graph segments can be represented by differentiable functions.
Figure 2.28

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (a) points with vertical tangents and (b) points at which the function is not continuous.

EXAMPLE 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x.

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. So, it does not define y as a differentiable function of x. See Figure 2.28(a).

b. The graph of this equation is the unit circle, centered at (0, 0). The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(b).

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point (1, 0), the slope of the graph is undefined. See Figure 2.28(c).

Try It | Exploration A | Exploration B

EXAMPLE 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$. See Figure 2.29.

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y} \qquad \text{Solve for } \frac{dy}{dx}.$$

So, at $\left(\sqrt{2}, -1/\sqrt{2}\right)$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

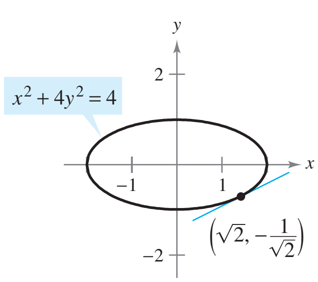


Figure 2.29

Editable Graph

Try It | Exploration A | Exploration B | Open Exploration

NOTE To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$.

EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$.

Solution

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

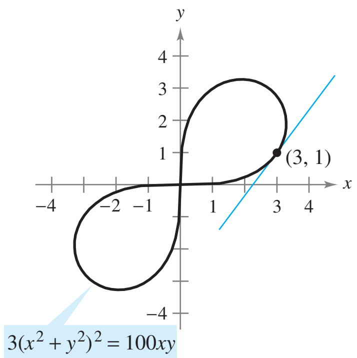
$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$



Lemniscate
Figure 2.30

At the point $(3, 1)$, the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 2.30. This graph is called a **lemniscate.**

Try It | Exploration A | Exploration B

EXAMPLE 6 Determining a Differentiable Function

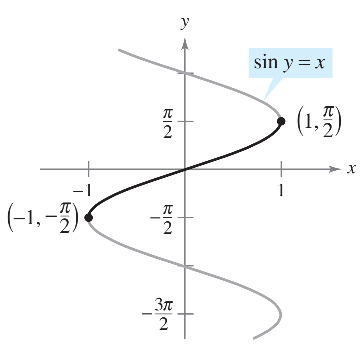
Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 2.31).

Solution

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$



The derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.
Figure 2.31

Editable Graph

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$, you should be able to write dy/dx explicitly as a function of x. To do this, you can use

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Try It | Exploration A

Isaac Barrow (1630–1677)

The graph in Figure 2.32 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.

MathBio

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Differentiating a second time with respect to x yields

$$\frac{d^2y}{dx^2} = -\frac{(y)(1) - (x)(dy/dx)}{y^2} \qquad \text{Quotient Rule}$$

$$= -\frac{y - (x)(-x/y)}{y^2} \qquad \text{Substitute } -x/y \text{ for } \frac{dy}{dx}.$$

$$= -\frac{y^2 + x^2}{y^3} \qquad \text{Simplify.}$$

$$= -\frac{25}{y^3}. \qquad \text{Substitute 25 for } x^2 + y^2.$$

Try It Exploration A

EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 2.32.

Solution By rewriting and differentiating implicitly, you obtain

$$x^4 + x^2y^2 - y^2 = 0$$

$$4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} = 0$$

$$2y(x^2 - 1)\frac{dy}{dx} = -2x(2x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{x(2x^2 + y^2)}{y(1 - x^2)}.$$

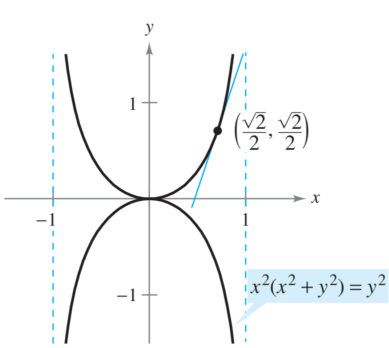
At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx} = \frac{(\sqrt{2}/2)[2(1/2) + (1/2)]}{(\sqrt{2}/2)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y = 3x - \sqrt{2}.$$



The kappa curve
Figure 2.32

Try It Exploration A

Exercises for Section 2.5

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1–16, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 36$
2. $x^2 - y^2 = 16$
3. $x^{1/2} + y^{1/2} = 9$
4. $x^3 + y^3 = 8$
5. $x^3 - xy + y^2 = 4$
6. $x^2y + y^2x = -2$
7. $x^3y^3 - y = x$
8. $\sqrt{xy} = x - 2y$
9. $x^3 - 3x^2y + 2xy^2 = 12$
10. $2 \sin x \cos y = 1$
11. $\sin x + 2 \cos 2y = 1$
12. $(\sin \pi x + \cos \pi y)^2 = 2$
13. $\sin x = x(1 + \tan y)$
14. $\cot y = x - y$
15. $y = \sin(xy)$
16. $x = \sec \dfrac{1}{y}$

In Exercises 17–20, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx and show that the result is equivalent to that of part (c).

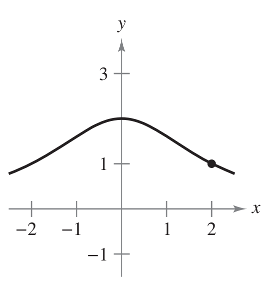
17. $x^2 + y^2 = 16$
18. $x^2 + y^2 - 4x + 6y + 9 = 0$
19. $9x^2 + 16y^2 = 144$
20. $9y^2 - x^2 = 9$

In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

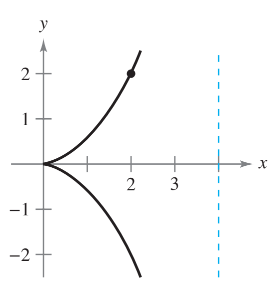
21. $xy = 4, \quad (-4, -1)$
22. $x^2 - y^3 = 0, \quad (1, 1)$
23. $y^2 = \dfrac{x^2 - 4}{x^2 + 4}, \quad (2, 0)$
24. $(x + y)^3 = x^3 + y^3, \quad (-1, 1)$
25. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$
26. $x^3 + y^3 = 4xy + 1, \quad (2, 1)$
27. $\tan(x + y) = x, \quad (0, 0)$
28. $x \cos y = 1, \quad \left(2, \dfrac{\pi}{3}\right)$

Famous Curves **In Exercises 29–32, find the slope of the tangent line to the graph at the given point.**

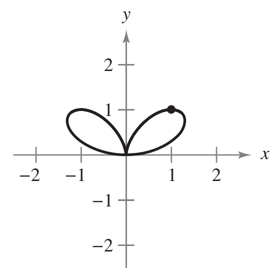
29. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



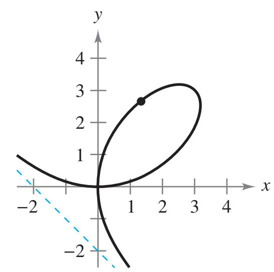
30. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



31. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$

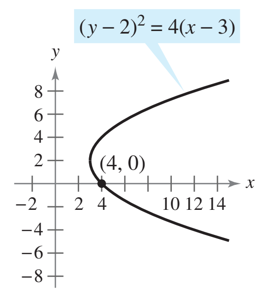


32. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$

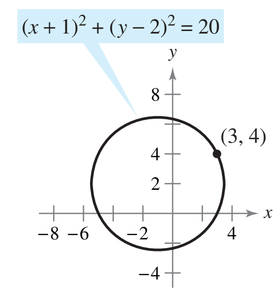


Famous Curves **In Exercises 33–40, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, select the MathGraph button.**

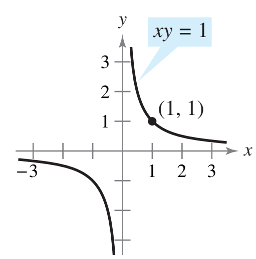
33. Parabola



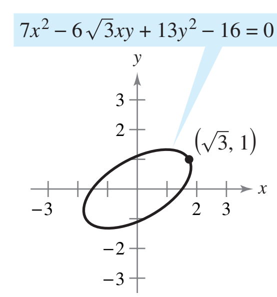
34. Circle



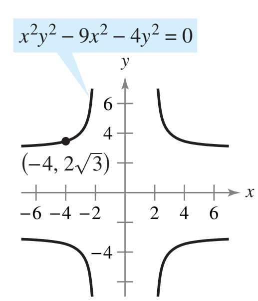
35. Rotated hyperbola



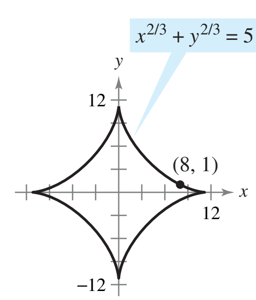
36. Rotated ellipse



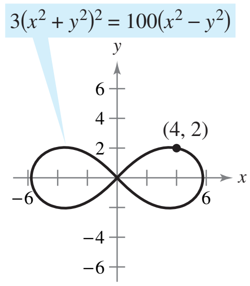
37. Cruciform



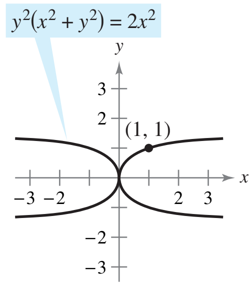
38. Astroid



39. Lemniscate



40. Kappa curve



41. (a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

42. (a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

In Exercises 43 and 44, find dy/dx implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that y is a differentiable function of x. Write dy/dx as a function of x.

43. $\tan y = x$ **44.** $\cos y = x$

In Exercises 45–50, find d^2y/dx^2 in terms of x and y.

45. $x^2 + y^2 = 36$ **46.** $x^2y^2 - 2x = 3$

47. $x^2 - y^2 = 16$ **48.** $1 - xy = x - y$

49. $y^2 = x^3$ **50.** $y^2 = 4x$

In Exercises 51 and 52, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.

51. $\sqrt{x} + \sqrt{y} = 4$, $(9, 1)$ **52.** $y^2 = \frac{x-1}{x^2+1}$, $\left(2, \frac{\sqrt{5}}{5}\right)$

In Exercises 53 and 54, find equations for the tangent line and normal line to the circle at the given points. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.

53. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$

54. $x^2 + y^2 = 9$
$(0, 3), (2, \sqrt{5})$

55. Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

56. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line.

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

Orthogonal Trajectories **In Exercises 59–62, use a graphing utility to sketch the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection their tangent lines are perpendicular to each other.]**

59. $2x^2 + y^2 = 6$
$y^2 = 4x$

60. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

61. $x + y = 0$
$x = \sin y$

62. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 63 and 64, verify that the two families of curves are orthogonal where C and K are real numbers. Use a graphing utility to graph the two families for two values of C and two values of K.**

63. $xy = C$, $x^2 - y^2 = K$ **64.** $x^2 + y^2 = C^2$, $y = Kx$

In Exercises 65–68, differentiate (a) with respect to x (y is a function of x) and (b) with respect to t (x and y are functions of t).

65. $2y^2 - 3x^4 = 0$ **66.** $x^2 - 3xy^2 + y^3 = 10$

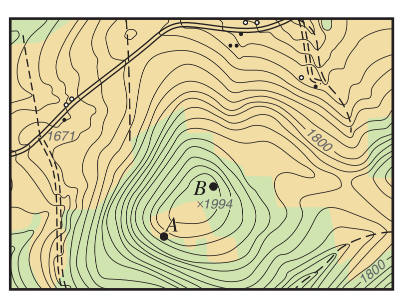
67. $\cos \pi y - 3 \sin \pi x = 1$ **68.** $4 \sin x \cos y = 1$

Writing About Concepts

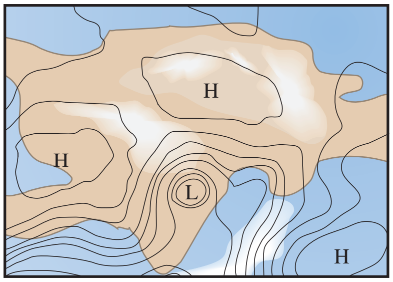
69. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

70. In your own words, state the guidelines for implicit differentiation.

71. ***Orthogonal Trajectories*** The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the graph, select the MathGraph button.



72. ***Weather Map*** The weather map shows several *isobars*—curves that represent areas of constant air pressure. Three high pressures H and one low pressure L are shown on the map. Given that wind speed is greatest along the orthogonal trajectories of the isobars, use the map to determine the areas having high wind speed.



73. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

74. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

75. Prove (Theorem 2.3) that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

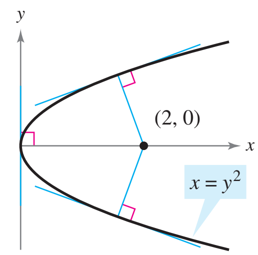
for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

76. ***Slope*** Find all points on the circle $x^2 + y^2 = 25$ where the slope is $\frac{3}{4}$.

77. ***Horizontal Tangent*** Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has a horizontal tangent.

78. ***Tangent Lines*** Find equations of both tangent lines to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that passes through the point $(4, 0)$.

79. ***Normals to a Parabola*** The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of x_0 are two of the normal lines perpendicular to each other?



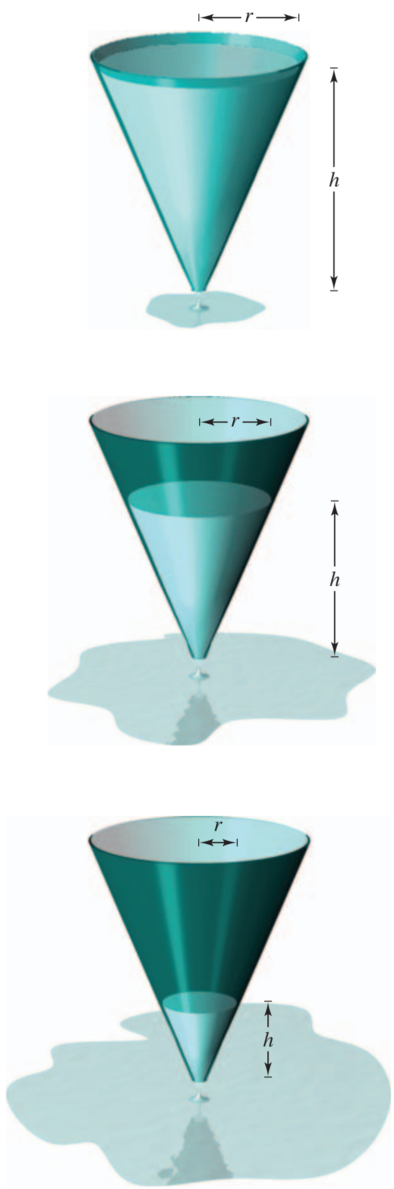
80. ***Normal Lines*** (a) Find an equation of the normal line to the ellipse

$$\frac{x^2}{32} + \frac{y^2}{8} = 1$$

at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

Section 2.6 Related Rates

- **Find a related rate.**
- **Use related rates to solve real-life problems.**



Volume is related to radius and height.
Figure 2.33

Animation

Finding Related Rates

You have seen how the Chain Rule can be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank (see Figure 2.33), the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h \qquad \text{Original equation}$$

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t.$$

$$= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r.

EXPLORATION

Finding a Related Rate In the conical tank shown in Figure 2.33, suppose that the height is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of r and h? Explain.

EXAMPLE 1 Two Rates That Are Related

Suppose x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$y = x^2 + 3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \text{Chain Rule}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

Try It   Exploration A

FOR FURTHER INFORMATION To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*.

MathArticle

Problem Solving with Related Rates

In Example 1, you were *given* an equation that related the variables x and y and were asked to find the rate of change of y when $x = 1$.

Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2$ when $x = 1$

Find: $\frac{dy}{dt}$ when $x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.

EXAMPLE 2 Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.34. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Total area increases as the outer radius increases.
Figure 2.34

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

Equation: $A = \pi r^2$

Given rate: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$ Differentiate with respect to t.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$ Chain Rule

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi$$ Substitute 4 for r and 1 for dr/dt.

When the radius is 4 feet, the area is changing at a rate of 8π square feet per second.

Try It | Exploration A | Video | Video

Guidelines For Solving Related-Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

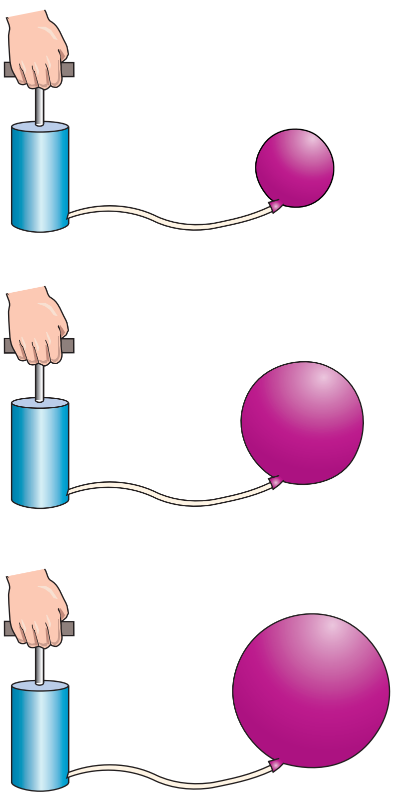
NOTE When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\frac{dx}{dt} = 50$ when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\frac{dV}{dt} = 10 \text{ m}^3/\text{hr}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min

EXAMPLE 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.



Inflating a balloon
Figure 2.35

Animation

Solution Let V be the volume of the balloon and let r be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

Given rate: $\frac{dV}{dt} = \frac{9}{2}$ (constant rate)

Find: $\frac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \frac{4}{3}\pi r^3$ Volume of a sphere

Differentiating both sides of the equation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$ Differentiate with respect to t.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right).$$ Solve for dr/dt.

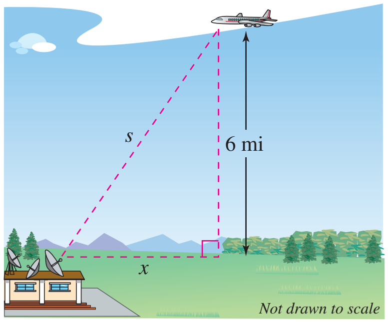
Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

Try It | Exploration A | Video

In Example 3, note that the volume is increasing at a *constant* rate but the radius is increasing at a *variable* rate. Just because two rates are related does not mean that they are proportional. In this particular case, the radius is growing more and more slowly as t increases. Do you see why?

EXAMPLE 4 The Speed of an Airplane Tracked by Radar



An airplane is flying at an altitude of 6 miles, s miles from the station.
Figure 2.36

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Solution Let x be the horizontal distance from the station, as shown in Figure 2.36. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $ds/dt = -400$ when $s = 10$

Find: dx/dt when $s = 10$ and $x = 8$

You can find the velocity of the plane as shown.

Equation: $x^2 + 6^2 = s^2$ — Pythagorean Theorem

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$ Differentiate with respect to t.

$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right)$$ Solve for dx/dt.

$$\frac{dx}{dt} = \frac{10}{8}(-400)$$ Substitute for s, x, and ds/dt.

$$= -500 \text{ miles per hour}$$ Simplify.

Because the velocity is -500 miles per hour, the *speed* is 500 miles per hour.

Try It | Exploration A | Open Exploration

EXAMPLE 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 2.37 at 10 seconds after lift-off.

Solution Let θ be the angle of elevation, as shown in Figure 2.37. When $t = 10$, the height s of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

Given rate: $ds/dt = 100t$ = velocity of rocket

Find: $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 2.37, you can relate s and θ by the equation $\tan\theta = s/2000$.

Equation: $\tan\theta = \dfrac{s}{2000}$ — See Figure 2.37.

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right)$$ Differentiate with respect to t.

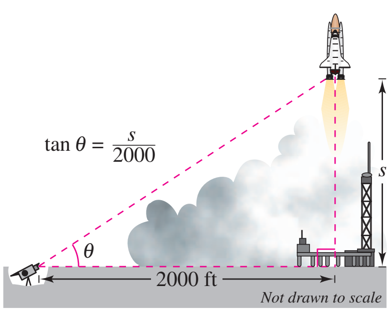
$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000}$$ Substitute $100t$ for ds/dt.

$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2 \frac{100t}{2000}$$ $\cos\theta = 2000/\sqrt{s^2 + 2000^2}$

When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

So, when $t = 10$, θ is changing at a rate of $\frac{2}{29}$ radian per second.



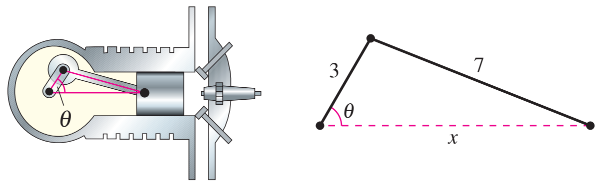
A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad.
Figure 2.37

Animation

Try It | Exploration A

EXAMPLE 6 The Velocity of a Piston

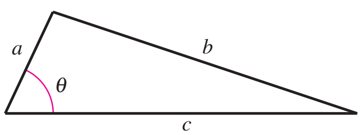
In the engine shown in Figure 2.38, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



The velocity of a piston is related to the angle of the crankshaft.
Figure 2.38

Animation

Solution Label the distances as shown in Figure 2.38. Because a complete revolution corresponds to 2π radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minute.



Law of Cosines:
$b^2 = a^2 + c^2 - 2ac\cos\theta$
Figure 2.39

Given rate: $\dfrac{d\theta}{dt} = 400\pi$ (constant rate)

Find: $\dfrac{dx}{dt}$ when $\theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (Figure 2.39) to find an equation that relates x and θ.

Equation:

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\theta$$

$$0 = 2x\frac{dx}{dt} - 6\left(-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right)$$

$$(6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)$$

When $\theta = \pi/3$, you can solve for x as shown.

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3}$$

$$49 = 9 + x^2 - 6x\left(\frac{1}{2}\right)$$

$$0 = x^2 - 3x - 40$$

$$0 = (x - 8)(x + 5)$$

$$x = 8$$ Choose positive solution.

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)\left(\sqrt{3}/2\right)}{6(1/2) - 16}(400\pi)$$

$$= \frac{9600\pi\sqrt{3}}{-13}$$

$$\approx -4018 \text{ inches per minute.}$$

Try It   Exploration A

NOTE Note that the velocity in Example 6 is negative because x represents a distance that is decreasing.

Exercises for Section 2.6

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

	Equation	*Find*	*Given*
1.	$y = \sqrt{x}$	(a) $\frac{dy}{dt}$ when $x = 4$	$\frac{dx}{dt} = 3$
		(b) $\frac{dx}{dt}$ when $x = 25$	$\frac{dy}{dt} = 2$
2.	$y = 2(x^2 - 3x)$	(a) $\frac{dy}{dt}$ when $x = 3$	$\frac{dx}{dt} = 2$
		(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = 5$
3.	$xy = 4$	(a) $\frac{dy}{dt}$ when $x = 8$	$\frac{dx}{dt} = 10$
		(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = -6$
4.	$x^2 + y^2 = 25$	(a) $\frac{dy}{dt}$ when $x = 3, y = 4$	$\frac{dx}{dt} = 8$
		(b) $\frac{dx}{dt}$ when $x = 4, y = 3$	$\frac{dy}{dt} = -2$

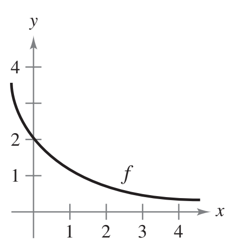
In Exercises 5–8, a point is moving along the graph of the given function such that dx/dt is 2 centimeters per second. Find dy/dt for the given values of x.

5.	$y = x^2 + 1$	(a) $x = -1$	(b) $x = 0$	(c) $x = 1$
6.	$y = \frac{1}{1 + x^2}$	(a) $x = -2$	(b) $x = 0$	(c) $x = 2$
7.	$y = \tan x$	(a) $x = -\frac{\pi}{3}$	(b) $x = -\frac{\pi}{4}$	(c) $x = 0$
8.	$y = \sin x$	(a) $x = \frac{\pi}{6}$	(b) $x = \frac{\pi}{4}$	(c) $x = \frac{\pi}{3}$

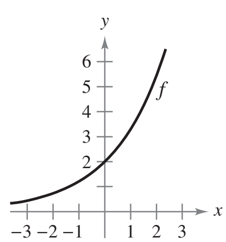
Writing About Concepts

In Exercises 9 and 10, using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.

9.



10.



11. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

Writing About Concepts *(continued)*

12. In your own words, state the guidelines for solving related-rate problems.

13. Find the rate of change of the distance between the origi and a moving point on the graph of $y = x^2 + 1$ if $dx/dt =$ 2 centimeters per second.

14. Find the rate of change of the distance between the origi and a moving point on the graph of $y = \sin x$ if $dx/dt =$ 2 centimeters per second.

15. ***Area*** The radius r of a circle is increasing at a rate of 3 centimeters per minute. Find the rates of change of the area when (a) $r = 6$ centimeters and (b) $r = 24$ centimeters.

16. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

17. ***Area*** The included angle of the two sides of constant equa length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$

(b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

18. ***Volume*** The radius r of a sphere is increasing at a rate of 2 inches per minute.

(a) Find the rate of change of the volume when $r = 6$ inches and $r = 24$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

19. ***Volume*** A spherical balloon is inflated with gas at the rate o 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

20. ***Volume*** All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

21. ***Surface Area*** The conditions are the same as in Exercise 20. Determine how fast the *surface area* is changing when each edge is (a) 1 centimeter and (b) 10 centimeters.

22. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

23. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet pe minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

24. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

25. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

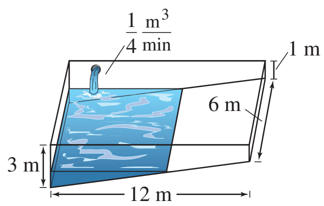


Figure for 25

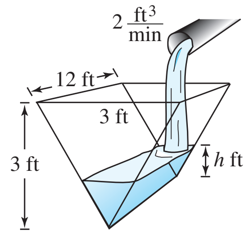


Figure for 26

26. ***Depth*** A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when h is 1 foot deep?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

27. ***Moving Ladder*** A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

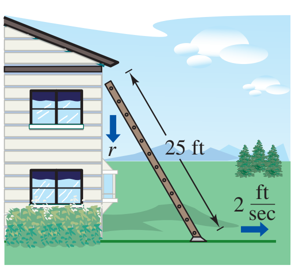


Figure for 27

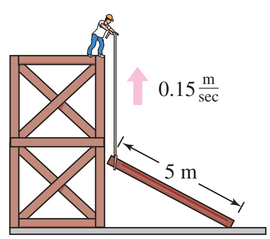


Figure for 28

FOR FURTHER INFORMATION For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*.

MathArticle

28. ***Construction*** A construction worker pulls a five-meter plan up the side of a building under construction by means of a rop tied to one end of the plank (see figure). Assume the opposit end of the plank follows a path perpendicular to the wall of th building and the worker pulls the rope at a rate of 0.15 mete per second. How fast is the end of the plank sliding along th ground when it is 2.5 meters from the wall of the building?

29. ***Construction*** A winch at the top of a 12-meter building pull a pipe of the same length to a vertical position, as shown in th figure. The winch pulls in rope at a rate of -0.2 meter pe second. Find the rate of vertical change and the rate of hori zontal change at the end of the pipe when $y = 6$.

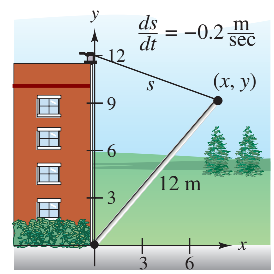


Figure for 29

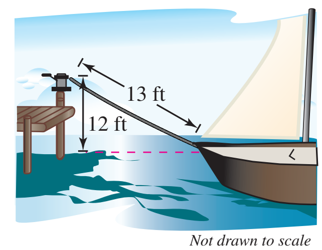


Figure for 30

30. ***Boating*** A boat is pulled into a dock by means of a winch 1 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second Determine the speed of the boat when there is 13 feet o rope out. What happens to the speed of the boat as it get closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet pe second. Determine the speed at which the winch pulls i rope when there is a total of 13 feet of rope out. Wha happens to the speed at which the winch pulls in rope as th boat gets closer to the dock?

31. ***Air Traffic Control*** An air traffic controller spots two plane at the same altitude converging on a point as they fly at righ angles to each other (see figure). One plane is 150 miles fron the point moving at 450 miles per hour. The other plane is 20 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing

(b) How much time does the air traffic controller have to ge one of the planes on a different flight path?

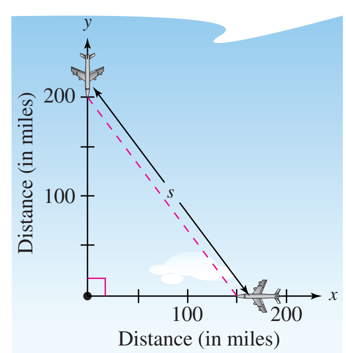


Figure for 31

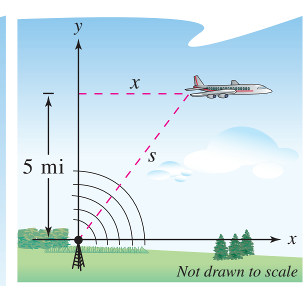


Figure for 32

32. ***Air Traffic Control*** An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure on previous page). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

33. ***Sports*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 28 feet per second is 30 feet from third base. At what rate is the player's distance s from home plate changing?

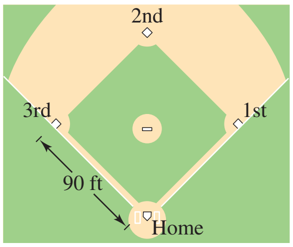


Figure for 33 and 34

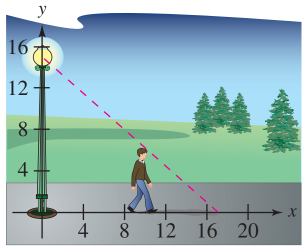


Figure for 35

34. ***Sports*** For the baseball diamond in Exercise 33, suppose the player is running from first to second at a speed of 28 feet per second. Find the rate at which the distance from home plate is changing when the player is 30 feet from second base.

35. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

36. ***Shadow Length*** Repeat Exercise 35 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

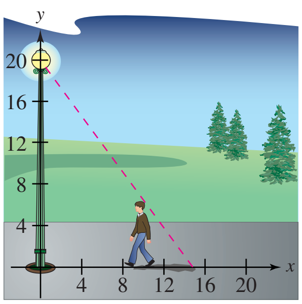


Figure for 36

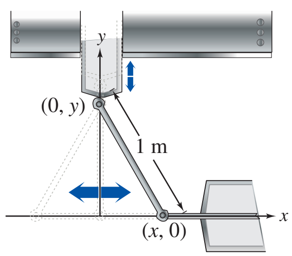


Figure for 37

37. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

38. ***Machine Design*** Repeat Exercise 37 for a position functio of $x(t) = \frac{3}{5}\sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

39. ***Evaporation*** As a spherical raindrop falls, it reaches a laye of dry air and begins to evaporate at a rate that is proportiona to its surface area ($S = 4\pi r^2$). Show that the radius of th raindrop decreases at a constant rate.

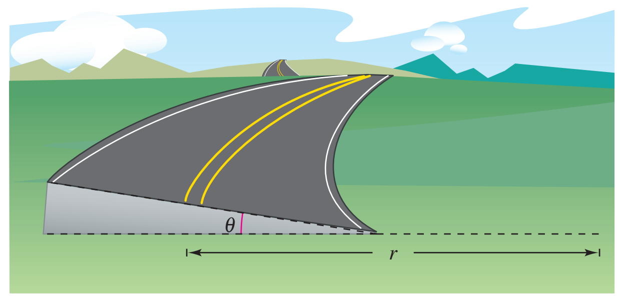
40. ***Electricity*** The combined electrical resistance R of R_1 and R_2 connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 ar increasing at rates of 1 and 1.5 ohms per second, respectively. A what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms

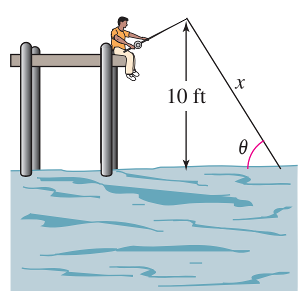
41. ***Adiabatic Expansion*** When a certain polyatomic ga undergoes adiabatic expansion, its pressure p and volume V satisfy the equation $pV^{1.3} = k$, where k is a constant. Find th relationship between the related rates dp/dt and dV/dt.

42. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone t overcome the centrifugal force, the road is banked at an angl of magnitude θ from the horizontal (see figure). The bankin angle must satisfy the equation $rg \tan\theta = v^2$, where v is th velocity of the cars and $g = 32$ feet per second per second i the acceleration due to gravity. Find the relationship betwee the related rates dv/dt and $d\theta/dt$.

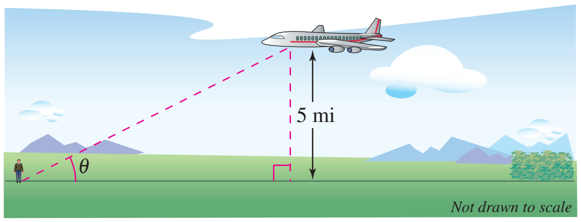


43. ***Angle of Elevation*** A balloon rises at a rate of 3 meters pe second from a point on the ground 30 meters from an observe Find the rate of change of the angle of elevation of the balloo from the observer when the balloon is 30 meters above th ground.

44. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot pe second from a point 10 feet above the water (see figure). A what rate is the angle between the line and the water changin when there is a total of 25 feet of line out?



45. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.



46. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the line perpendicular from the light to the wall?

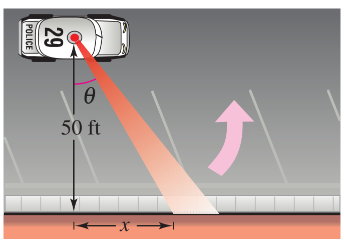


Figure for 46

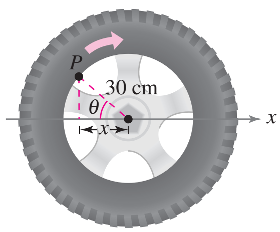


Figure for 47

47. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

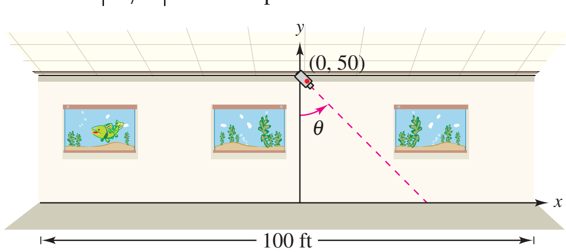
(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

48. ***Flight Control*** An airplane is flying in still air with an airspeed of 240 miles per hour. If it is climbing at an angle of 22°, find the rate at which it is gaining altitude.

49. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.



50. ***Think About It*** Describe the relationship between the rate o change of y and the rate of change of x in each expression Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 51 and 52, find the acceleration o the specified object. (*Hint:* Recall that if a variable is changin at a constant rate, its acceleration is zero.)**

51. Find the acceleration of the top of the ladder described i Exercise 27 when the base of the ladder is 7 feet from the wall

52. Find the acceleration of the boat in Exercise 30(a) when ther is a total of 13 feet of rope out.

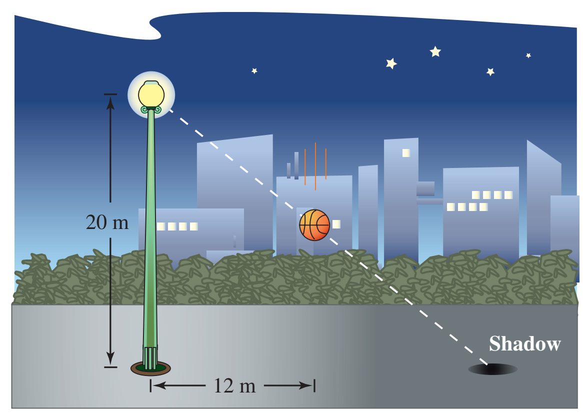
53. ***Modeling Data*** The table shows the numbers (in millions) o single women (never married) s and married women m in th civilian work force in the United States for the years 199 through 2001. *(Source: U.S. Bureau of Labor Statistics)*

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001
s	15.0	15.3	15.5	15.8	16.5	17.1	17.6	17.8	18.0
m	32.0	32.9	33.4	33.6	33.8	33.9	34.4	34.6	34.7

(a) Use the regression capabilities of a graphing utility to fin a model of the form $m(s) = as^3 + bs^2 + cs + d$ for th data, where t is the time in years, with $t = 3$ correspondin to 1993.

(b) Find dm/dt. Then use the model to estimate dm/dt fo $t = 10$ if it is predicted that the number of single women i the work force will increase at the rate of 0.75 millio per year.

54. ***Moving Shadow*** A ball is dropped from a height o 20 meters, 12 meters away from the top of a 20-meter lamppos (see figure). The ball's shadow, caused by the light at the top o the lamppost, is moving along the level ground. How fast is th shadow moving 1 second after the ball is released *(Submitted by Dennis Gittinger, St. Philips College, Sa Antonio, TX)*



Review Exercises for Chapter 2

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

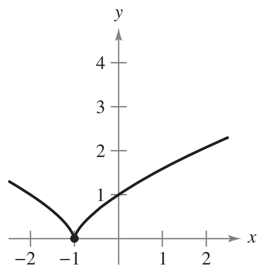
In Exercises 1–4, find the derivative of the function by using the definition of the derivative.

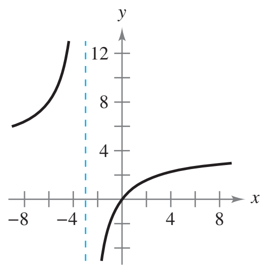
1. $f(x) = x^2 - 2x + 3$ **2.** $f(x) = \sqrt{x} + 1$

3. $f(x) = \dfrac{x+1}{x-1}$ **4.** $f(x) = \dfrac{2}{x}$

In Exercises 5 and 6, describe the x-values at which f is differentiable.

5. $f(x) = (x+1)^{2/3}$ **6.** $f(x) = \dfrac{4x}{x+3}$





7. Sketch the graph of $f(x) = 4 - |x - 2|$.

(a) Is f continuous at $x = 2$?

(b) Is f differentiable at $x = 2$? Explain.

8. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2, & x < -2 \\ 1 - 4x - x^2, & x \geq -2. \end{cases}$

(a) Is f continuous at $x = -2$?

(b) Is f differentiable at $x = -2$? Explain.

In Exercises 9 and 10, find the slope of the tangent line to the graph of the function at the given point.

9. $g(x) = \dfrac{2}{3}x^2 - \dfrac{x}{6}, \quad \left(-1, \dfrac{5}{6}\right)$

10. $h(x) = \dfrac{3x}{8} - 2x^2, \quad \left(-2, -\dfrac{35}{4}\right)$

In Exercises 11 and 12, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

11. $f(x) = x^3 - 1, \quad (-1, -2)$ **12.** $f(x) = \dfrac{2}{x+1}, \quad (0, 2)$

In Exercises 13 and 14, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

13. $g(x) = x^2(x - 1), \quad c = 2$ **14.** $f(x) = \dfrac{1}{x+1}, \quad c = 2$

In Exercises 15–30, find the derivative of the function.

15. $y = 25$ **16.** $y = -12$

17. $f(x) = x^8$ **18.** $g(x) = x^{12}$

19. $h(t) = 3t^4$ **20.** $f(t) = -8t^5$

21. $f(x) = x^3 - 3x^2$ **22.** $g(s) = 4s^4 - 5s^2$

23. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$ **24.** $f(x) = x^{1/2} - x^{-1/2}$

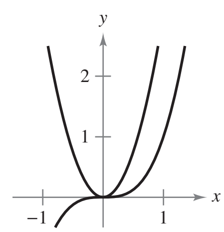
25. $g(t) = \dfrac{2}{3t^2}$ **26.** $h(x) = \dfrac{2}{(3x)^2}$

27. $f(\theta) = 2\theta - 3\sin\theta$ **28.** $g(\alpha) = 4\cos\alpha + 6$

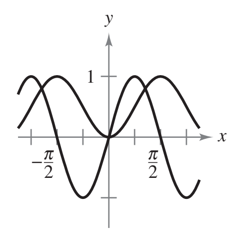
29. $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$ **30.** $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

Writing **In Exercises 31 and 32, the figure shows the graphs o a function and its derivative. Label the graphs as f or f' anc write a short paragraph stating the criteria used in making thi selection. To print an enlarged copy of the graph, select thi MathGraph button.**

31.



32.



33. ***Vibrating String*** When a guitar string is plucked, it vibrate: with a frequency of $F = 200\sqrt{T}$, where F is measured in vibra tions per second and the tension T is measured in pounds. Finc the rates of change of F when (a) $T = 4$ and (b) $T = 9$.

34. ***Vertical Motion*** A ball is dropped from a height of 100 feet One second later, another ball is dropped from a height of 7: feet. Which ball hits the ground first?

35. ***Vertical Motion*** To estimate the height of a building, a weigh is dropped from the top of the building into a pool at grounc level. How high is the building if the splash is seen 9.2 second: after the weight is dropped?

36. ***Vertical Motion*** A bomb is dropped from an airplane at an alti tude of 14,400 feet. How long will it take for the bomb to reacl the ground? (Because of the motion of the plane, the fall will no be vertical, but the time will be the same as that for a vertica fall.) The plane is moving at 600 miles per hour. How far will thi bomb move horizontally after it is released from the plane?

37. ***Projectile Motion*** A ball thrown follows a path described by $y = x - 0.02x^2$.

(a) Sketch a graph of the path.

(b) Find the total horizontal distance the ball is thrown.

(c) At what x-value does the ball reach its maximum height? (Use the symmetry of the path.)

(d) Find an equation that gives the instantaneous rate of chang of the height of the ball with respect to the horizonta change. Evaluate the equation at $x = 0, 10, 25, 30,$ and 50

(e) What is the instantaneous rate of change of the height whei the ball reaches its maximum height?

38. ***Projectile Motion*** The path of a projectile thrown at an angle of 45° with level ground is

$$y = x - \frac{32}{v_0^2}(x^2)$$

where the initial velocity is v_0 feet per second.

(a) Find the x-coordinate of the point where the projectile strikes the ground. Use the symmetry of the path of the projectile to locate the x-coordinate of the point where the projectile reaches its maximum height.

(b) What is the instantaneous rate of change of the height when the projectile is at its maximum height?

(c) Show that doubling the initial velocity of the projectile multiplies both the maximum height and the range by a factor of 4.

(d) Find the maximum height and range of a projectile thrown with an initial velocity of 70 feet per second. Use a graphing utility to graph the path of the projectile.

39. ***Horizontal Motion*** The position function of a particle moving along the x-axis is

$$x(t) = t^2 - 3t + 2 \quad \text{for} \quad -\infty < t < \infty.$$

(a) Find the velocity of the particle.

(b) Find the open t-interval(s) in which the particle is moving to the left.

(c) Find the position of the particle when the velocity is 0.

(d) Find the speed of the particle when the position is 0.

40. ***Modeling Data*** The speed of a car in miles per hour and the stopping distance in feet are recorded in the table.

Speed, x	20	30	40	50	60
Stopping Distance, y	25	55	105	188	300

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use a graphing utility to graph dy/dx.

(d) Use the model to approximate the stopping distance at a speed of 65 miles per hour.

(e) Use the graphs in parts (b) and (c) to explain the change in stopping distance as the speed increases.

In Exercises 41–54, find the derivative of the function.

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

42. $g(x) = (x^3 - 3x)(x + 2)$

43. $h(x) = \sqrt{x}\sin x$ **44.** $f(t) = t^3 \cos t$

45. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$ **46.** $f(x) = \dfrac{6x - 5}{x^2 + 1}$

47. $f(x) = \dfrac{1}{4 - 3x^2}$ **48.** $f(x) = \dfrac{9}{3x^2 - 2x}$

49. $y = \dfrac{x^2}{\cos x}$ **50.** $y = \dfrac{\sin x}{x^2}$

51. $y = 3x^2 \sec x$ **52.** $y = 2x - x^2 \tan x$

53. $y = x\cos x - \sin x$ **54.** $g(x) = 3x \sin x + x^2 \cos x$

In Exercises 55–58, find an equation of the tangent line to the graph of f at the given point.

55. $f(x) = \dfrac{2x^3 - 1}{x^2}$, $(1, 1)$ **56.** $f(x) = \dfrac{x + 1}{x - 1}$, $\left(\dfrac{1}{2}, -3\right)$

57. $f(x) = -x\tan x$, $(0, 0)$ **58.** $f(x) = \dfrac{1 + \sin x}{1 - \sin x}$, $(\pi, 1)$

59. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 4$.

60. ***Acceleration*** An automobile's velocity starting from rest is

$$v(t) = \frac{90t}{4t + 10}$$

where v is measured in feet per second. Find the vehicle's velocity and acceleration at each of the following times.

(a) 1 second (b) 5 seconds (c) 10 seconds

In Exercises 61–64, find the second derivative of the function.

61. $g(t) = t^3 - 3t + 2$ **62.** $f(x) = 12\sqrt[4]{x}$

63. $f(\theta) = 3\tan\theta$ **64.** $h(t) = 4\sin t - 5\cos t$

In Exercises 65 and 66, show that the function satisfies the equation.

Function	*Equation*
65. $y = 2\sin x + 3\cos x$	$y'' + y = 0$
66. $y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

In Exercises 67–78, find the derivative of the function.

67. $h(x) = \left(\dfrac{x - 3}{x^2 + 1}\right)^2$ **68.** $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$ **70.** $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$

71. $y = 3\cos(3x + 1)$ **72.** $y = 1 - \cos 2x + 2\cos^2 x$

73. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$ **74.** $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

75. $y = \dfrac{2}{3}\sin^{3/2} x - \dfrac{2}{7}\sin^{7/2} x$ **76.** $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

77. $y = \dfrac{\sin \pi x}{x + 2}$ **78.** $y = \dfrac{\cos(x - 1)}{x - 1}$

In Exercises 79–82, find the derivative of the function at the given point.

79. $f(x) = \sqrt{1 - x^3}$, $(-2, 3)$

80. $f(x) = \sqrt[3]{x^2 - 1}$, $(3, 2)$

81. $y = \frac{1}{2}\csc 2x, \quad \left(\frac{\pi}{4}, \frac{1}{2}\right)$

82. $y = \csc 3x + \cot 3x, \quad \left(\frac{\pi}{6}, 1\right)$

In Exercises 83–86, use a computer algebra system to find the derivative of the function. Use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

83. $g(x) = \frac{2x}{\sqrt{x+1}}$ **84.** $f(x) = [(x-2)(x+4)]^2$

85. $f(t) = \sqrt{t+1}\sqrt[3]{t+1}$ **86.** $y = \sqrt{3x}(x+2)^3$

In Exercises 87–90, (a) use a computer algebra system to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the point, and (c) graph the function and its tangent line on the same set of coordinate axes.

87. $f(t) = t^2(t-1)^5, \quad (2, 4)$

88. $g(x) = x\sqrt{x^2+1}, \quad \left(3, 3\sqrt{10}\right)$

89. $y = \tan\sqrt{1-x}, \quad \left(-2, \tan\sqrt{3}\right)$

90. $y = 2\csc^3\left(\sqrt{x}\right), \quad (1, 2\csc^3 1)$

In Exercises 91–94, find the second derivative of the function.

91. $y = 2x^2 + \sin 2x$ **92.** $y = \frac{1}{x} + \tan x$

93. $f(x) = \cot x$ **94.** $y = \sin^2 x$

In Exercises 95–98, use a computer algebra system to find the second derivative of the function.

95. $f(t) = \frac{t}{(1-t)^2}$ **96.** $g(x) = \frac{6x-5}{x^2+1}$

97. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$ **98.** $h(x) = x\sqrt{x^2-1}$

99. ***Refrigeration*** The temperature T of food put in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

100. ***Fluid Flow*** The emergent velocity v of a liquid flowing from a hole in the bottom of a tank is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet per second per second) and h is the depth of the liquid in the tank. Find the rate of change of v with respect to h when (a) $h = 9$ and (b) $h = 4$. (Note that $g = +32$ feet per second per second. The sign of g depends on how a problem is modeled. In this case, letting g be negative would produce an imaginary value for v.)

In Exercises 101–106, use implicit differentiation to find dy/dx.

101. $x^2 + 3xy + y^3 = 10$ **102.** $x^2 + 9y^2 - 4x + 3y = 0$

103. $y\sqrt{x} - x\sqrt{y} = 16$ **104.** $y^2 = (x-y)(x^2+y)$

105. $x \sin y = y \cos x$ **106.** $\cos(x+y) = x$

In Exercises 107 and 108, find the equations of the tangent lin and the normal line to the graph of the equation at the give point. Use a graphing utility to graph the equation, the tangen line, and the normal line.

107. $x^2 + y^2 = 20, \quad (2, 4)$ **108.** $x^2 - y^2 = 16, \quad (5, 3)$

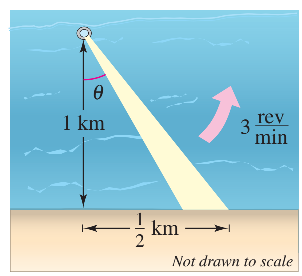
109. A point moves along the curve $y = \sqrt{x}$ in such a way that th y-value is increasing at a rate of 2 units per second. At wha rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

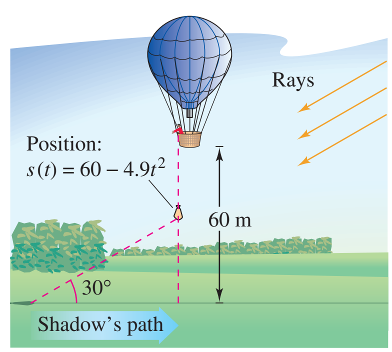
110. ***Surface Area*** The edges of a cube are expanding at a rat of 5 centimeters per second. How fast is the surface are changing when each edge is 4.5 centimeters?

111. ***Depth*** The cross section of a five-meter trough is an isosce les trapezoid with a two-meter lower base, a three-meter uppe base, and an altitude of 2 meters. Water is running into th trough at a rate of 1 cubic meter per minute. How fast is th water level rising when the water is 1 meter deep?

112. ***Linear and Angular Velocity*** A rotating beacon is locate 1 kilometer off a straight shoreline (see figure). If the beaco rotates at a rate of 3 revolutions per minute, how fast (i kilometers per hour) does the beam of light appear to b moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?



113. ***Moving Shadow*** A sandbag is dropped from a balloon at height of 60 meters when the angle of elevation to the sun i 30° (see figure). Find the rate at which the shadow of the sand bag is traveling along the ground when the sandbag is at height of 35 meters. [*Hint:* The position of the sandbag i given by $s(t) = 60 - 4.9t^2$.]



P.S. Problem Solving

The symbol [graphing icon] indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

1. Consider the graph of the parabola $y = x^2$.

(a) Find the radius r of the largest possible circle centered on the y-axis that is tangent to the parabola at the origin, as show in the figure. This circle is called the **circle of curvature** (see Section 12.5). Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

(b) Find the center $(0, b)$ of the circle of radius 1 centered on the y-axis that is tangent to the parabola at two points, as shown in the figure. Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

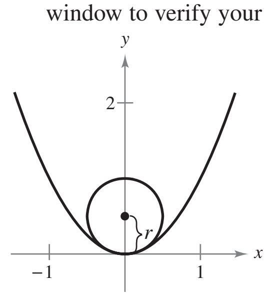


Figure for 1(a)

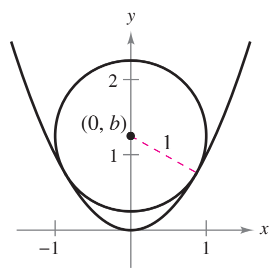


Figure for 1(b)

2. Graph the two parabolas $y = x^2$ and $y = -x^2 + 2x - 5$ in the same coordinate plane. Find equations of the two lines simultaneously tangent to both parabolas.

3. (a) Find the polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.

(b) Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree **Taylor polynomial** of $f(x) = \cos x$ at $x = 0$.

(c) Complete the table comparing the values of f and P_2. What do you observe?

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$							
$P_2(x)$							

(d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

4. (a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

(b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The normal line is perpendicular to the tangent line.) Where does this line intersect the parabola a second time?

(c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.

(d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

5. Find a third-degree polynomial $p(x)$ that is tangent to the line $y = 14x - 13$ at the point $(1, 1)$, and tangent to the line $y = -2x - 5$ at the point $(-1, -3)$.

6. Find a function of the form $f(x) = a + b\cos cx$ that is tangent to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

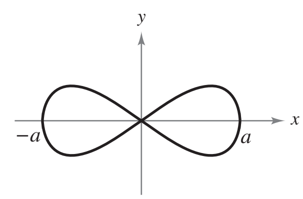
$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

7. The graph of the **eight curve,**

$x^4 = a^2(x^2 - y^2),\ a \neq 0,$

is shown below.



(a) Explain how you could use a graphing utility to graph this curve.

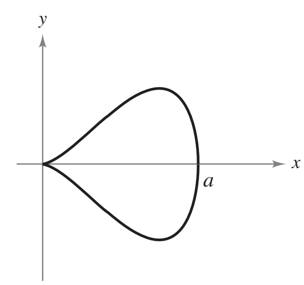
(b) Use a graphing utility to graph the curve for various values of the constant a. Describe how a affects the shape of the curve.

(c) Determine the points on the curve where the tangent line is horizontal.

8. The graph of the **pear-shaped quartic,**

$b^2y^2 = x^3(a - x),\ a, b > 0,$

is shown below.



(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constants a and b. Describe how a and b affect the shape of the curve.

(c) Determine the points on the curve where the tangent line is horizontal.

9. A man 6 feet tall walks at a rate of 5 feet per second toward a streetlight that is 30 feet high (see figure). The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

(a) Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

(d) Determine how fast the tip of the shadow is moving as a function of x, the distance between the man and the street light. Discuss the continuity of this shadow speed function.

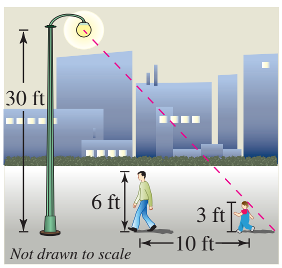


Figure for 9

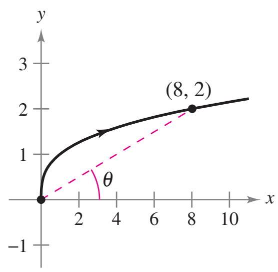


Figure for 10

10. A particle is moving along the graph of $y = \sqrt[3]{x}$ (see figure). When $x = 8$, the y-component of its position is increasing at the rate of 1 centimeter per second.

(a) How fast is the x-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination θ changing at this moment?

11. Let L be a differentiable function for all x. Prove that if $L(a + b) = L(a) + L(b)$ for all a and b, then $L'(x) = L'(0)$ for all x. What does the graph of L look like?

12. Let E be a function satisfying $E(0) = E'(0) = 1$. Prove that if $E(a + b) = E(a)E(b)$ for all a and b, then E is differentiable and $E'(x) = E(x)$ for all x. Find an example of a function satisfying $E(a + b) = E(a)E(b)$.

13. The fundamental limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ assumes that x is measured in radians. What happens if you assume that x is measured in degrees instead of radians?

(a) Set your calculator to *degree* mode and complete the table.

z **(in degrees)**	0.1	0.01	0.0001
$\frac{\sin z}{z}$			

(b) Use the table to estimate

$$\lim_{z\to 0} \frac{\sin z}{z}$$

for z in degrees. What is the exact value of this limit? (*Hint*: $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find

$$\frac{d}{dz} \sin z$$

for z in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate

$$\frac{d}{dz} S(z).$$

(e) Explain why calculus is made easier by using radians instead of degrees.

14. An astronaut standing on the moon throws a rock into the air. The height of the rock is

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the height of the rock at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

15. If a is the acceleration of an object, the *jerk* j is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of j.

(b) Find j for the slowing vehicle in Exercise 117 in Section 2.3 and interpret the result.

(c) The figure shows the graph of the position, velocity, acceleration, and jerk functions of a vehicle. Identify each graph and explain your reasoning.