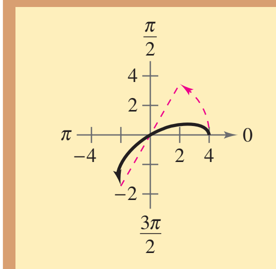
Chapter 9

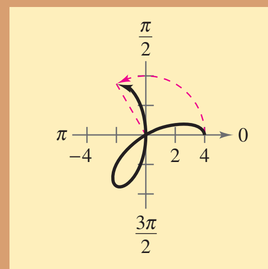
Topics in Analytic Geometry

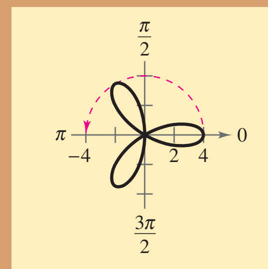
Selected Applications

Analytic geometry concepts have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Earthquake, Exercise 35, page 667
- Suspension Bridge, Exercise 93, page 669
- Architecture, Exercises 47–49, page 678
- Satellite Orbit, Exercise 54, page 679
- Navigation, Exercise 46, page 688
- Projectile Motion, Exercises 55–58, page 706
- Planetary Motion, Exercises 49–55, page 727
- Sports, Exercises 95–98, page 732







Conics are used to represent many real-life phenomena such as reflectors used in flashlights, orbits of planets, and navigation. In Chapter 9, you will learn how to write and graph equations of conics in rectangular and polar coordinates. You will also learn how to graph other polar equations and curves represented by parametric equations.

AP/Wide World Photos

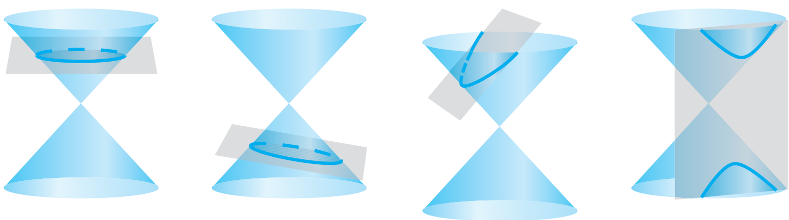
Satellites are used to monitor weather patterns, collect scientific data, and assist in navigation. Satellites orbit Earth in elliptical paths.

9.1 Circles and Parabolas

Conics

Conic sections were discovered during the classical Greek period, 600 to 300 B.C. The early Greek studies were largely concerned with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 9.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 9.2.



Circle *Ellipse* *Parabola* *Hyperbola*

Figure 9.1 *Basic Conics*

Point *Line* *Two intersecting lines*

Figure 9.2 *Degenerate Conics*

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property. For example, the definition of a circle as *the collection of all points (x, y) that are equidistant from a fixed point (h, k)* leads to the standard equation of a circle

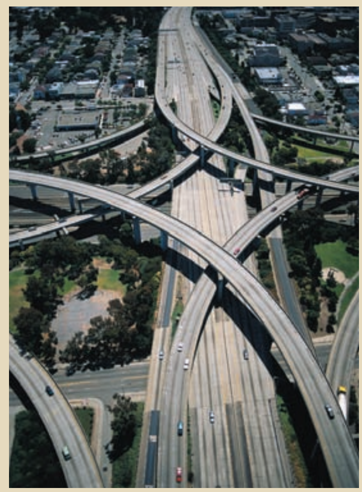
$$(x - h)^2 + (y - k)^2 = r^2. \quad \text{Equation of circle}$$

What you should learn

- Recognize a conic as the intersection of a plane and a double-napped cone.
- Write equations of circles in standard form.
- Write equations of parabolas in standard form.
- Use the reflective property of parabolas to solve real-life problems.

Why you should learn it

Parabolas can be used to model and solve many types of real-life problems. For instance, in Exercise 95 on page 669, a parabola is used to design the entrance ramp for a highway.



Circles

The definition of a circle as a locus of points is a more general definition of a circle as it applies to conics.

Definition of a Circle

A **circle** is the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k), called the **center** of the circle. (See Figure 9.3.) The distance r between the center and any point (x, y) on the circle is the **radius.**

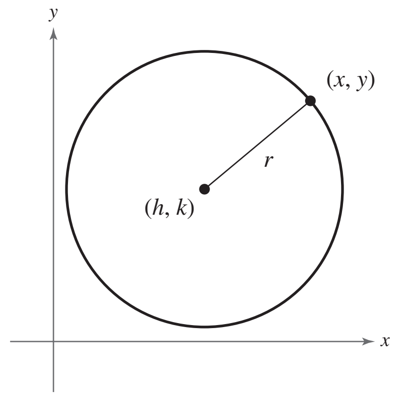


Figure 9.3

The Distance Formula can be used to obtain an equation of a circle whose center is (h, k) and whose radius is r.

$$\sqrt{(x-h)^2 + (y-k)^2} = r \qquad \text{Distance Formula}$$

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Square each side.}$$

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x-h)^2 + (y-k)^2 = r^2.$$

The point (h, k) is the center of the circle, and the positive number r is the radius of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2.$$

Example 1 Finding the Standard Equation of a Circle

The point $(1, 4)$ is on a circle whose center is at $(-2, -3)$, as shown in Figure 9.4. Write the standard form of the equation of the circle.

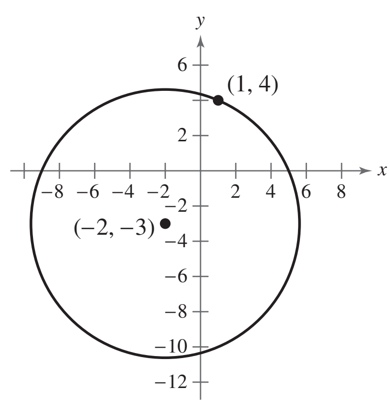


Figure 9.4

Solution

The radius of the circle is the distance between $(-2, -3)$ and $(1, 4)$.

$$r = \sqrt{[1-(-2)]^2 + [4-(-3)]^2} \qquad \text{Use Distance Formula.}$$

$$= \sqrt{3^2 + 7^2} \qquad \text{Simplify.}$$

$$= \sqrt{58} \qquad \text{Simplify.}$$

The equation of the circle with center $(h, k) = (-2, -3)$ and radius $r = \sqrt{58}$ is

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Standard form}$$

$$[x-(-2)]^2 + [y-(-3)]^2 = (\sqrt{58})^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x+2)^2 + (y+3)^2 = 58. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 3.

Example 2 Sketching a Circle

Sketch the circle given by the equation

$$x^2 - 6x + y^2 - 2y + 6 = 0$$

and identify its center and radius.

Solution

Begin by writing the equation in standard form.

$$x^2 - 6x + y^2 - 2y + 6 = 0 \qquad \text{Write original equation.}$$

$$(x^2 - 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1 \qquad \text{Complete the squares.}$$

$$(x - 3)^2 + (y - 1)^2 = 4 \qquad \text{Write in standard form.}$$

In this form, you can see that the graph is a circle whose center is the point (3, 1) and whose radius is $r = \sqrt{4} = 2$. Plot several points that are two units from the center. The points (5, 1), (3, 3), (1, 1), and (3, −1) are convenient. Draw a circle that passes through the four points, as shown in Figure 9.5.

CHECKPOINT Now try Exercise 23.

Prerequisite Skills

To use a graphing utility to graph a circle, review Appendix B.2.

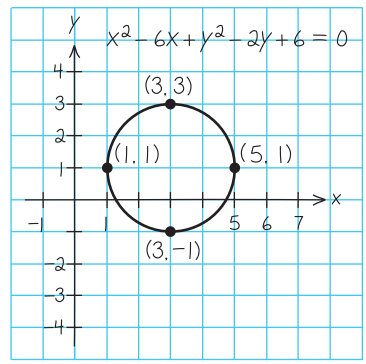


Figure 9.5

Example 3 Finding the Intercepts of a Circle

Find the x- and y-intercepts of the graph of the circle given by the equation

$$(x - 4)^2 + (y - 2)^2 = 16.$$

Solution

To find any x-intercepts, let $y = 0$. To find any y-intercepts, let $x = 0$.

x-intercepts:

$$(x - 4)^2 + (0 - 2)^2 = 16 \qquad \text{Substitute 0 for } y.$$

$$(x - 4)^2 = 12 \qquad \text{Simplify.}$$

$$x - 4 = \pm\sqrt{12} \qquad \text{Take square root of each side.}$$

$$x = 4 \pm 2\sqrt{3} \qquad \text{Add 4 to each side.}$$

y-intercepts:

$$(0 - 4)^2 + (y - 2)^2 = 16 \qquad \text{Substitute 0 for } x.$$

$$(y - 2)^2 = 0 \qquad \text{Simplify.}$$

$$y - 2 = 0 \qquad \text{Take square root of each side.}$$

$$y = 2 \qquad \text{Add 2 to each side.}$$

So the x-intercepts are $\left(4 + 2\sqrt{3}, 0\right)$ and $\left(4 - 2\sqrt{3}, 0\right)$, and the y-intercept is (0, 2), as shown in Figure 9.6.

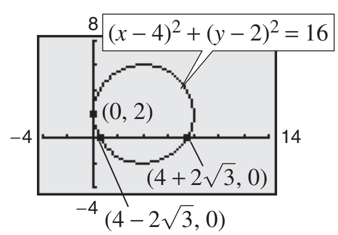


Figure 9.6

CHECKPOINT Now try Exercise 29.

Parabolas

In Section 2.1, you learned that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

> **Definition of a Parabola**
>
> A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. (See Figure 9.7.) The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola.

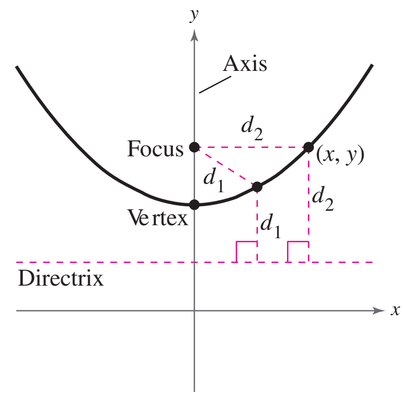
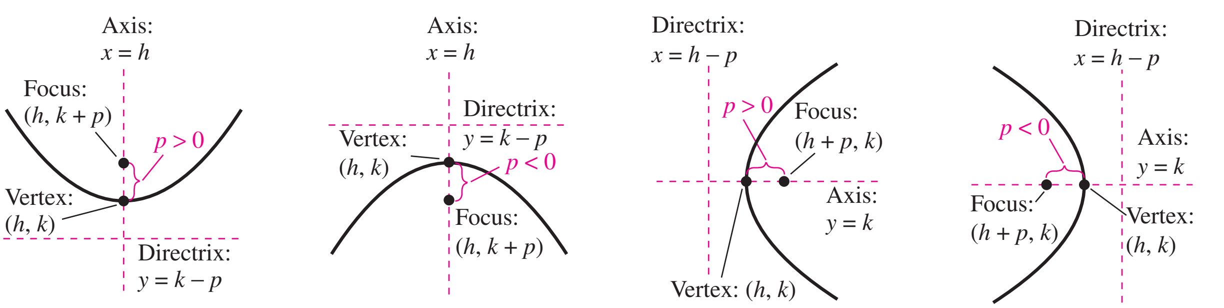


Figure 9.7

Note in Figure 9.7 that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form of the equation of a parabola** whose directrix is parallel to the x-axis or to the y-axis.

> **Standard Equation of a Parabola** (See the proof on page 737.)
>
> The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.
>
> $(x - h)^2 = 4p(y - k),\ p \neq 0$ — Vertical axis; directrix: $y = k - p$
>
> $(y - k)^2 = 4p(x - h),\ p \neq 0$ — Horizontal axis; directrix: $x = h - p$
>
> The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.
>
> $x^2 = 4py$ — Vertical axis
>
> $y^2 = 4px$ — Horizontal axis
>
> See Figure 9.8.



$(x - h)^2 = 4p(y - k)$
(a) Vertical axis: $p > 0$

$(x - h)^2 = 4p(y - k)$
(b) Vertical axis: $p < 0$

$(y - k)^2 = 4p(x - h)$
(c) Horizontal axis: $p > 0$

$(y - k)^2 = 4p(x - h)$
(d) Horizontal axis: $p < 0$

Figure 9.8

Example 4 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex at the origin and focus $(0, 4)$.

Solution

Because the axis of the parabola is vertical, passing through $(0, 0)$ and $(0, 4)$, consider the equation

$$x^2 = 4py.$$

Because the focus is $p = 4$ units from the vertex, the equation is

$$x^2 = 4(4)y$$
$$x^2 = 16y.$$

You can obtain the more common quadratic form as follows.

$$x^2 = 16y \qquad \text{Write original equation.}$$
$$\tfrac{1}{16}x^2 = y \qquad \text{Multiply each side by } \tfrac{1}{16}.$$

Use a graphing utility to confirm that the graph is a parabola, as shown in Figure 9.9.

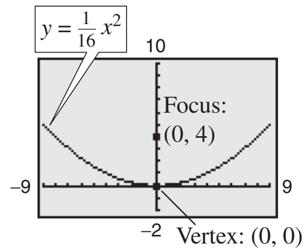


Figure 9.9

CHECKPOINT Now try Exercise 45.

Example 5 Finding the Focus of a Parabola

Find the focus of the parabola given by $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$.

Solution

To find the focus, convert to standard form by completing the square.

$$y = -\tfrac{1}{2}x^2 - x + \tfrac{1}{2} \qquad \text{Write original equation.}$$
$$-2y = x^2 + 2x - 1 \qquad \text{Multiply each side by } -2.$$
$$1 - 2y = x^2 + 2x \qquad \text{Add 1 to each side.}$$
$$1 + 1 - 2y = x^2 + 2x + 1 \qquad \text{Complete the square.}$$
$$2 - 2y = x^2 + 2x + 1 \qquad \text{Combine like terms.}$$
$$-2(y - 1) = (x + 1)^2 \qquad \text{Write in standard form.}$$

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

you can conclude that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$. Because p is negative, the parabola opens downward, as shown in Figure 9.10. Therefore, the focus of the parabola is

$$(h, k + p) = \left(-1, \tfrac{1}{2}\right). \qquad \text{Focus}$$

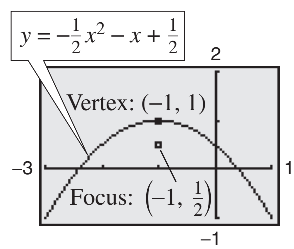


Figure 9.10

CHECKPOINT Now try Exercise 63.

Example 6 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex $(1, 0)$ and focus at $(2, 0)$.

Solution

Because the axis of the parabola is horizontal, passing through $(1, 0)$ and $(2, 0)$, consider the equation

$$(y - k)^2 = 4p(x - h)$$

where $h = 1$, $k = 0$, and $p = 2 - 1 = 1$. So, the standard form is

$$(y - 0)^2 = 4(1)(x - 1) \quad \Rightarrow \quad y^2 = 4(x - 1).$$

The parabola is shown in Figure 9.11.

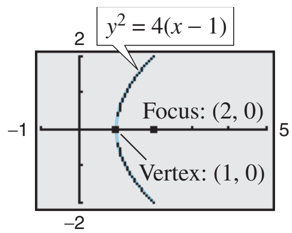


Figure 9.11

CHECKPOINT Now try Exercise 77.

TECHNOLOGY TIP Use a graphing utility to confirm the equation found in Example 6. To do this, it helps to graph the equation using two separate equations: $y_1 = \sqrt{4(x - 1)}$ (upper part) and $y_2 = -\sqrt{4(x - 1)}$ (lower part). Note that when you graph conics using two separate equations, your graphing utility may not connect the two parts. This is because some graphing utilities are limited in their resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

Reflective Property of Parabolas

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 9.12.

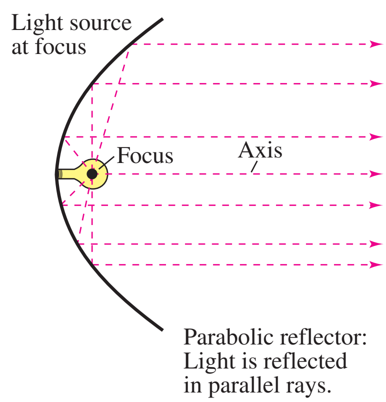


Figure 9.12

A line is **tangent** to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point. Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.

> **Reflective Property of a Parabola**
>
> The tangent line to a parabola at a point P makes equal angles with the following two lines (see Figure 9.13).
>
> 1. The line passing through P and the focus
> 2. The axis of the parabola

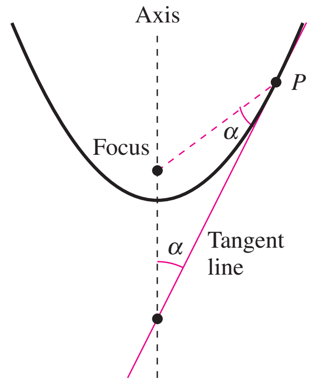


Figure 9.13

Example 7 Finding the Tangent Line at a Point on a Parabola

Find the equation of the tangent line to the parabola given by $y = x^2$ at the point $(1, 1)$.

Solution

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$, as shown in Figure 9.14. You can find the y-intercept $(0, b)$ of the tangent line by equating the lengths of the two sides of the isosceles triangle shown in Figure 9.14:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1 - 0)^2 + \left(1 - \frac{1}{4}\right)^2} = \frac{5}{4}.$$

Note that $d_1 = \frac{1}{4} - b$ rather than $b - \frac{1}{4}$. The order of subtraction for the distance is important because the distance must be positive. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$

CHECKPOINT Now try Exercise 85.

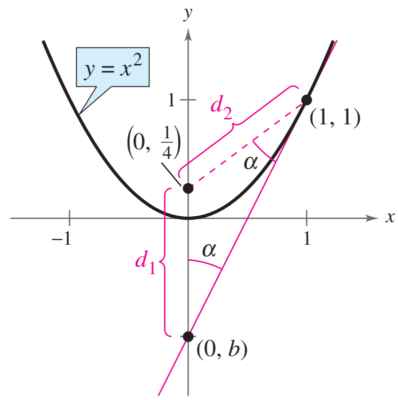


Figure 9.14

TECHNOLOGY TIP Try using a graphing utility to confirm the result of Example 7. By graphing

$$y_1 = x^2 \qquad \text{and} \qquad y_2 = 2x - 1$$

in the same viewing window, you should be able to see that the line touches the parabola at the point $(1, 1)$.

9.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A _______ is the intersection of a plane and a double-napped cone.
2. A collection of points satisfying a geometric property can also be referred to as a _______ of points.
3. A _______ is the set of all points (x, y) in a plane that are equidistant from a fixed point, called the _______ .
4. A _______ is the set of all points (x, y) in a plane that are equidistant from a fixed line, called the _______ , and a fixed point, called the _______ , not on the line.
5. The _______ of a parabola is the midpoint between the focus and the directrix.
6. The line that passes through the focus and vertex of a parabola is called the _______ of the parabola.
7. A line is _______ to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point.

In Exercises 1–6, find the standard form of the equation of the circle with the given characteristics.

1. Center at origin; radius: $\sqrt{18}$
2. Center at origin; radius: $4\sqrt{2}$
3. Center: $(3, 7)$; point on circle: $(1, 0)$
4. Center: $(6, -3)$; point on circle: $(-2, 4)$
5. Center: $(-3, -1)$; diameter: $2\sqrt{7}$
6. Center: $(5, -6)$; diameter: $4\sqrt{3}$

In Exercises 7–12, identify the center and radius of the circle.

7. $x^2 + y^2 = 49$
8. $x^2 + y^2 = 1$
9. $(x + 2)^2 + (y - 7)^2 = 16$
10. $(x + 9)^2 + (y + 1)^2 = 36$
11. $(x - 1)^2 + y^2 = 15$
12. $x^2 + (y + 12)^2 = 24$

In Exercises 13–20, write the equation of the circle in standard form. Then identify its center and radius.

13. $\frac{1}{4}x^2 + \frac{1}{4}y^2 = 1$
14. $\frac{1}{9}x^2 + \frac{1}{9}y^2 = 1$
15. $\frac{4}{3}x^2 + \frac{4}{3}y^2 = 1$
16. $\frac{9}{2}x^2 + \frac{9}{2}y^2 = 1$
17. $x^2 + y^2 - 2x + 6y + 9 = 0$
18. $x^2 + y^2 - 10x - 6y + 25 = 0$
19. $4x^2 + 4y^2 + 12x - 24y + 41 = 0$
20. $9x^2 + 9y^2 + 54x - 36y + 17 = 0$

In Exercises 21–28, sketch the circle. Identify its center and radius.

21. $x^2 = 16 - y^2$
22. $y^2 = 81 - x^2$
23. $x^2 + 4x + y^2 + 4y - 1 = 0$
24. $x^2 - 6x + y^2 + 6y + 14 = 0$
25. $x^2 - 14x + y^2 + 8y + 40 = 0$
26. $x^2 + 6x + y^2 - 12y + 41 = 0$
27. $x^2 + 2x + y^2 - 35 = 0$
28. $x^2 + y^2 + 10y + 9 = 0$

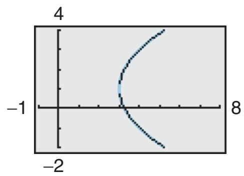
In Exercises 29–34, find the x- and y-intercepts of the graph of the circle.

29. $(x - 2)^2 + (y + 3)^2 = 9$
30. $(x + 5)^2 + (y - 4)^2 = 25$
31. $x^2 - 2x + y^2 - 6y - 27 = 0$
32. $x^2 + 8x + y^2 + 2y + 9 = 0$
33. $(x - 6)^2 + (y + 3)^2 = 16$
34. $(x + 7)^2 + (y - 8)^2 = 4$

35. ***Earthquake*** An earthquake was felt up to 81 miles from its epicenter. You were located 60 miles west and 45 miles south of the epicenter.
    (a) Let the epicenter be at the point $(0, 0)$. Find the standard equation that describes the outer boundary of the earthquake.
    (b) Would you have felt the earthquake?
    (c) Verify your answer to part (b) by graphing the equation of the outer boundary of the earthquake and plotting your location. How far were you from the outer boundary of the earthquake?

36. ***Landscaper*** A landscaper has installed a circular sprinkler system that covers an area of 1800 square feet.
    (a) Find the radius of the region covered by the sprinkler system. Round your answer to three decimal places.
    (b) If the landscaper wants to cover an area of 2400 square feet, how much longer does the radius need to be?

In Exercises 37–42, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)


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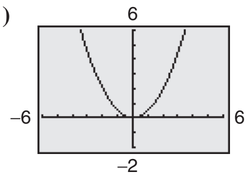

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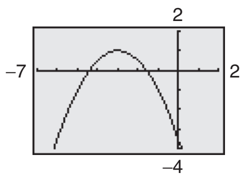

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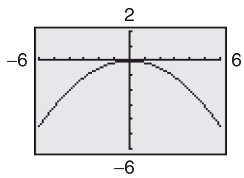

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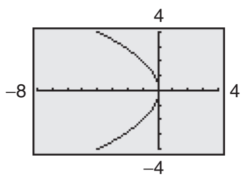

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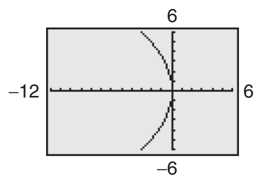


37. $y^2 = -4x$

38. $x^2 = 2y$

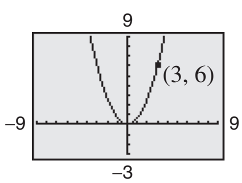
39. $x^2 = -8y$

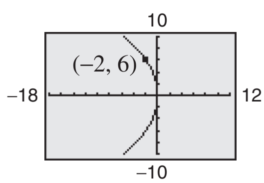
40. $y^2 = -12x$

41. $(y - 1)^2 = 4(x - 3)$

42. $(x + 3)^2 = -2(y - 1)$

In Exercises 43–54, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.

43.


44.


45. Focus: $\left(0, -\frac{3}{2}\right)$

46. Focus: $\left(\frac{5}{2}, 0\right)$

47. Focus: $(-2, 0)$

48. Focus: $(0, 1)$

49. Directrix: $y = -1$

50. Directrix: $y = 3$

51. Directrix: $x = 2$

52. Directrix: $x = -3$

53. Horizontal axis and passes through the point $(4, 6)$

54. Vertical axis and passes through the point $(-3, -3)$

In Exercises 55–72, find the vertex, focus, and directrix of the parabola and sketch its graph.

55. $y = \frac{1}{2}x^2$

56. $y = -4x^2$

57. $y^2 = -6x$

58. $y^2 = 3x$

59. $x^2 + 8y = 0$

60. $x + y^2 = 0$

61. $(x + 1)^2 + 8(y + 3) = 0$

62. $(x - 5) + (y + 4)^2 = 0$

63. $y^2 + 6y + 8x + 25 = 0$

64. $y^2 - 4y - 4x = 0$

65. $\left(x + \frac{3}{2}\right)^2 = 4(y - 2)$

66. $\left(x + \frac{1}{2}\right)^2 = 4(y - 1)$

67. $y = \frac{1}{4}(x^2 - 2x + 5)$

68. $x = \frac{1}{4}(y^2 + 2y + 33)$

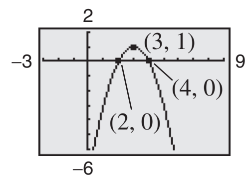
69. $x^2 + 4x + 6y - 2 = 0$

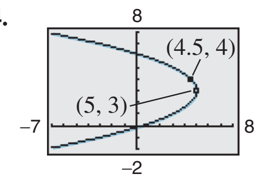
70. $x^2 - 2x + 8y + 9 = 0$

71. $y^2 + x + y = 0$

72. $y^2 - 4x - 4 = 0$

In Exercises 73–82, find the standard form of the equation of the parabola with the given characteristics.

73.


74.


75. Vertex: $(-2, 0)$; focus: $\left(-\frac{3}{2}, 0\right)$

76. Vertex: $(3, -3)$; focus: $\left(3, -\frac{9}{4}\right)$

77. Vertex: $(5, 2)$; focus: $(3, 2)$

78. Vertex: $(-1, 2)$; focus: $(-1, 0)$

79. Vertex: $(0, 4)$; directrix: $y = 2$

80. Vertex: $(-2, 1)$; directrix: $x = 1$

81. Focus: $(2, 2)$; directrix: $x = -2$

82. Focus: $(0, 0)$; directrix: $y = 4$

In Exercises 83 and 84, the equations of a parabola and a tangent line to the parabola are given. Use a graphing utility to graph both in the same viewing window. Determine the coordinates of the point of tangency.

	Parabola	*Tangent Line*
83.	$y^2 - 8x = 0$	$x - y + 2 = 0$
84.	$x^2 + 12y = 0$	$x + y - 3 = 0$

In Exercises 85–88, find an equation of the tangent line to the parabola at the given point and find the x-intercept of the line.

85. $x^2 = 2y$, $(4, 8)$

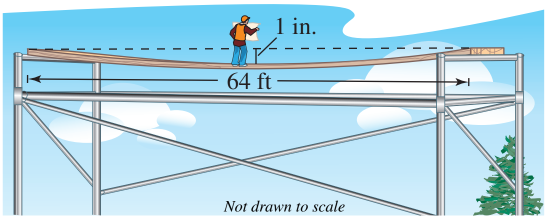
86. $x^2 = 2y$, $\left(-3, \frac{9}{2}\right)$

87. $y = -2x^2$, $(-1, -2)$

88. $y = -2x^2$, $(2, -8)$

89. ***Revenue*** The revenue R (in dollars) generated by the sale of x 32-inch televisions is modeled by $R = 375x - \frac{3}{2}x^2$. Use a graphing utility to graph the function and approximate the sales that will maximize revenue.

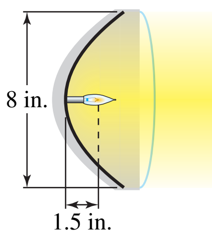
90. ***Beam Deflection*** A simply supported beam is 64 feet long and has a load at the center (see figure). The deflection (bending) of the beam at its center is 1 inch. The shape of the deflected beam is parabolic.



(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection equal to $\frac{1}{2}$ inch?

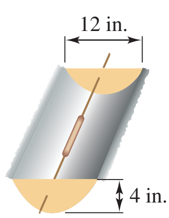
91. ***Automobile Headlight*** The filament of an automobile headlight is at the focus of a parabolic reflector, which sends light out in a straight beam (see figure).



(a) The filament of the headlight is 1.5 inches from the vertex. Find an equation for the cross section of the reflector.

(b) The reflector is 8 inches wide. Find the depth of the reflector.

92. ***Solar Cooker*** You want to make a solar hot dog cooker using aluminum foil-lined cardboard, shaped as a parabolic trough. The figure shows how to suspend the hot dog with a wire through the foci of the ends of the parabolic trough. The parabolic end pieces are 12 inches wide and 4 inches deep. How far from the bottom of the trough should the wire be inserted?



93. ***Suspension Bridge*** Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway midway between the towers.

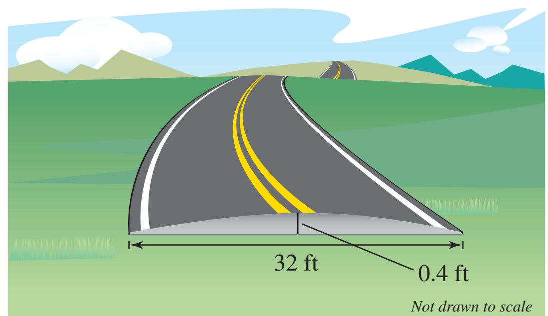
(a) Draw a sketch of the bridge. Locate the origin of a rectangular coordinate system at the center of the roadway. Label the coordinates of the known points.

(b) Write an equation that models the cables.

(c) Complete the table by finding the height y of the suspension cables over the roadway at a distance of x meters from the center of the bridge.

x	0	200	400	500	600
y					

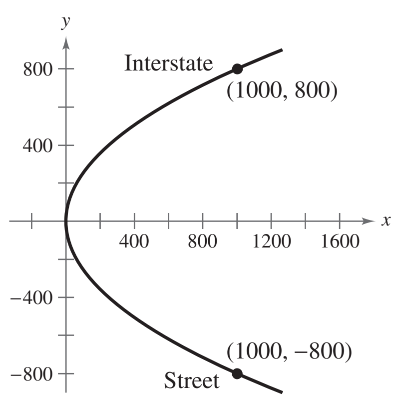
94. ***Road Design*** Roads are often designed with parabolic surfaces to allow rain to drain off. A particular road that is 32 feet wide is 0.4 foot higher in the center than it is on the sides (see figure).



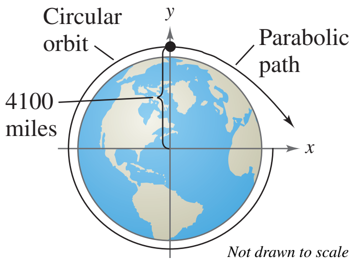
(a) Find an equation of the parabola that models the road surface. (Assume that the origin is at the center of the road.)

(b) How far from the center of the road is the road surface 0.1 foot lower than in the middle?

95. ***Highway Design*** Highway engineers design a parabolic curve for an entrance ramp from a straight street to an interstate highway (see figure). Find an equation of the parabola.



96. ***Satellite Orbit*** A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. If this velocity is multiplied by $\sqrt{2}$, the satellite will have the minimum velocity necessary to escape Earth's gravity, and it will follow a parabolic path with the center of Earth as the focus (see figure).



(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of Earth is 4000 miles).

97. ***Path of a Projectile*** The path of a softball is modeled by

$$-12.5(y - 7.125) = (x - 6.25)^2.$$

The coordinates x and y are measured in feet, with $x = 0$ corresponding to the position from which the ball was thrown.

(a) Use a graphing utility to graph the trajectory of the softball.

(b) Use the *zoom* and *trace* features of the graphing utility to approximate the highest point the ball reaches and the distance the ball travels.

98. ***Projectile Motion*** Consider the path of a projectile projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is $x^2 = -\frac{1}{16}v^2(y - s)$. In this model, air resistance is disregarded, y is the height (in feet) of the projectile, and x is the horizontal distance (in feet) the projectile travels. A ball is thrown from the top of a 75-foot tower with a velocity of 32 feet per second.

(a) Find the equation of the parabolic path.

(b) How far does the ball travel horizontally before striking the ground?

In Exercises 99–102, find an equation of the tangent line to the circle at the indicated point. Recall from geometry that the tangent line to a circle is perpendicular to the radius of the circle at the point of tangency.

	Circle	*Point*
99.	$x^2 + y^2 = 25$	$(3, -4)$
100.	$x^2 + y^2 = 169$	$(-5, 12)$
101.	$x^2 + y^2 = 12$	$(2, -2\sqrt{2})$
102.	$x^2 + y^2 = 24$	$(-2\sqrt{5}, 2)$

Synthesis

True or False? **In Exercises 103–108, determine whether the statement is true or false. Justify your answer.**

103. The equation $x^2 + (y + 5)^2 = 25$ represents a circle with its center at the origin and a radius of 5.

104. The graph of the equation $x^2 + y^2 = r^2$ will have x-intercepts $(\pm r, 0)$ and y-intercepts $(0, \pm r)$.

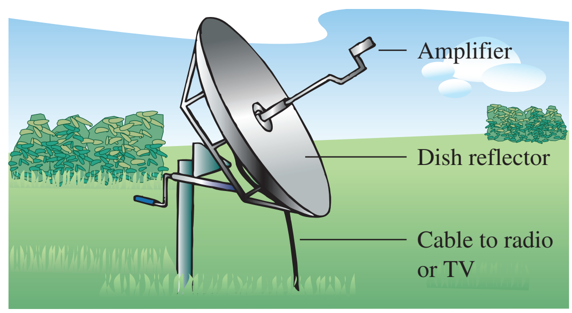
105. A circle is a degenerate conic.

106. It is possible for a parabola to intersect its directrix.

107. The point which lies on the graph of a parabola closest to its focus is the vertex of the parabola.

108. The directrix of the parabola $x^2 = y$ intersects, or is tangent to, the graph of the parabola at its vertex, $(0, 0)$.

109. ***Writing*** Cross sections of television antenna dishes are parabolic in shape (see figure). Write a paragraph describing why these dishes are parabolic. Include a graphical representation of your description.



110. ***Think About It*** The equation $x^2 + y^2 = 0$ is a degenerate conic. Sketch the graph of this equation and identify the degenerate conic. Describe the intersection of the plane with the double-napped cone for this particular conic.

Think About It **In Exercises 111 and 112, change the equation so that its graph matches the description.**

111. $(y - 3)^2 = 6(x + 1)$; upper half of parabola

112. $(y + 1)^2 = 2(x - 2)$; lower half of parabola

Skills Review

In Exercises 113–116, use a graphing utility to approximate any relative minimum or maximum values of the function.

113. $f(x) = 3x^3 - 4x + 2$

114. $f(x) = 2x^2 + 3x$

115. $f(x) = x^4 + 2x + 2$

116. $f(x) = x^5 - 3x - 1$

9.2 Ellipses

What you should learn

- Write equations of ellipses in standard form.
- Use properties of ellipses to model and solve real-life problems.
- Find eccentricities of ellipses.

Why you should learn it

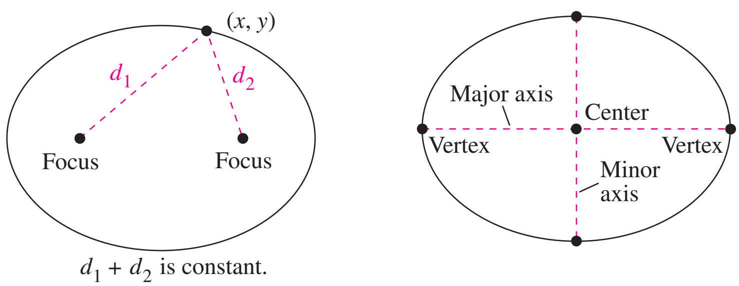
Ellipses can be used to model and solve many types of real-life problems. For instance, in Exercise 50 on page 678, an ellipse is used to model the floor of Statuary Hall, an elliptical room in the U.S. Capitol Building in Washington, D.C.

Introduction

The third type of conic is called an **ellipse.** It is defined as follows.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant. [See Figure 9.15(a).]



(a) (b)

Figure 9.15

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis.** [See Figure 9.15(b).]

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 9.16. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

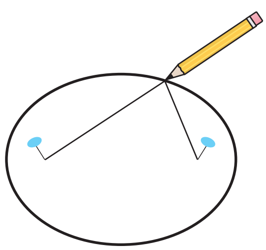
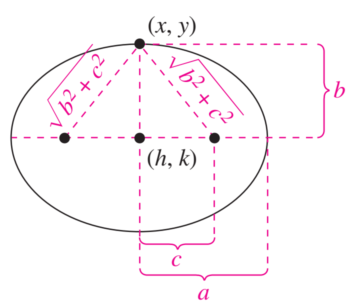


Figure 9.16



$$2\sqrt{b^2 + c^2} = 2a$$
$$b^2 + c^2 = a^2$$

Figure 9.17

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 9.17 with the following points: center, (h, k); vertices, $(h \pm a, k)$; foci, $(h \pm c, k)$. Note that the center is the midpoint of the segment joining the foci.

The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

or simply the length of the major axis.

Now, if you let (x, y) be *any* point on the ellipse, the sum of the distances between (x, y) and the two foci must also be $2a$. That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a.$$

Finally, in Figure 9.17, you can see that $b^2 = a^2 - c^2$, which implies that the equation of the ellipse is

$$b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis. Both results are summarized as follows.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

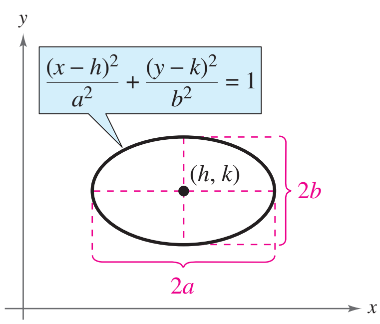
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{Major axis is vertical.}$$

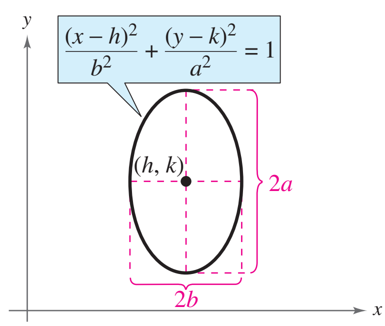
Exploration

On page 671 it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil. Try doing this. Vary the length of the string and the distance between the two thumbtacks. Explain how to obtain ellipses that are almost circular. Explain how to obtain ellipses that are long and narrow.

Figure 9.18 shows both the vertical and horizontal orientations for an ellipse.



Major axis is horizontal.



Major axis is vertical.

Figure 9.18

Example 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse having foci at (0, 1) and (4, 1) and a major axis of length 6, as shown in Figure 9.19.

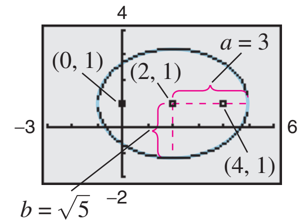


Figure 9.19

Solution

By the Midpoint Formula, the center of the ellipse is (2, 1) and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{9 - 4} = \sqrt{5}.$$

Because the major axis is horizontal, the standard equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{(\sqrt{5})^2} = 1.$$

CHECKPOINT Now try Exercise 35.

TECHNOLOGY SUPPORT

For instructions on how to use the *zoom* and *trace* features, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Example 2 Sketching an Ellipse

Sketch the ellipse given by $4x^2 + y^2 = 36$ and identify the center and vertices.

Algebraic Solution

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

The center of the ellipse is (0, 0). Because the denominator of the y^2-term is larger than the denominator of the x^2-term, you can conclude that the major axis is vertical. Moreover, because $a = 6$, the vertices are $(0, -6)$ and $(0, 6)$. Finally, because $b = 3$, the endpoints of the minor axis are $(-3, 0)$ and $(3, 0)$, as shown in Figure 9.20.

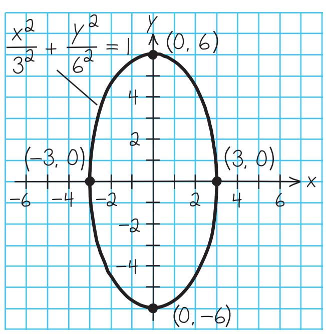


Figure 9.20

CHECKPOINT Now try Exercise 13.

Graphical Solution

Solve the equation of the ellipse for y as follows.

$$4x^2 + y^2 = 36$$
$$y^2 = 36 - 4x^2$$
$$y = \pm\sqrt{36 - 4x^2}$$

Then use a graphing utility to graph $y_1 = \sqrt{36 - 4x^2}$ and $y_2 = -\sqrt{36 - 4x^2}$ in the same viewing window. Be sure to use a square setting. From the graph in Figure 9.21, you can see that the major axis is vertical and its center is at the point (0, 0). You can use the *zoom* and *trace* features to approximate the vertices to be (0, 6) and $(0, -6)$.

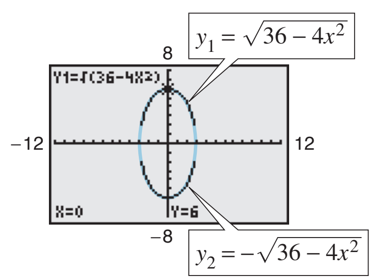


Figure 9.21

Example 3 Graphing an Ellipse

Graph the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Solution

Begin by writing the original equation in standard form. In the third step, note that 9 and 4 are added to *both* sides of the equation when completing the squares.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \quad \text{Write original equation.}$$

$$(x^2 + 6x + \boxed{}) + 4(y^2 - 2y + \boxed{}) = -9 \quad \text{Group terms and factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \quad \text{Write in completed square form.}$$

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1 \quad \text{Write in standard form.}$$

Now you see that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The graph of this ellipse is shown in Figure 9.22.

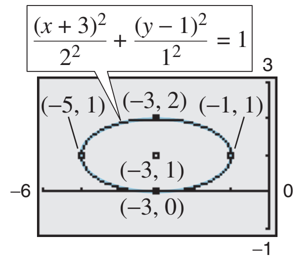


Figure 9.22

CHECKPOINT Now try Exercise 15.

Example 4 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution

By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \quad \text{Write original equation.}$$

$$4(x^2 - 2x + \boxed{}) + (y^2 + 4y + \boxed{}) = 8 \quad \text{Group terms and factor 4 out of } x\text{-terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4$$

$$4(x - 1)^2 + (y + 2)^2 = 16 \quad \text{Write in completed square form.}$$

$$\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{4^2} = 1 \quad \text{Write in standard form.}$$

So, the major axis is vertical, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

Therefore, you have the following.

Center: $(1, -2)$ Vertices: $(1, -6)$, $(1, 2)$ Foci: $(1, -2 - 2\sqrt{3})$, $(1, -2 + 2\sqrt{3})$

The graph of the ellipse is shown in Figure 9.23.

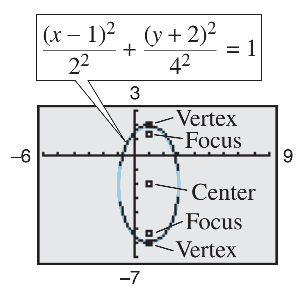


Figure 9.23

CHECKPOINT Now try Exercise 17.

TECHNOLOGY TIP

You can use a graphing utility to graph an ellipse by graphing the upper and lower portions in the same viewing window. For instance, to graph the ellipse in Example 3, first solve for y to obtain

$$y_1 = 1 + \sqrt{1 - \frac{(x + 3)^2}{4}}$$

and

$$y_2 = 1 - \sqrt{1 - \frac{(x + 3)^2}{4}}.$$

Use a viewing window in which $-6 \le x \le 0$ and $-1 \le y \le 3$. You should obtain the graph shown in Figure 9.22.

Application

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 5 investigates the elliptical orbit of the moon about Earth.

Example 5 An Application Involving an Elliptical Orbit

The moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in Figure 9.24. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and smallest distances (the *apogee* and *perigee*) from Earth's center to the moon's center.

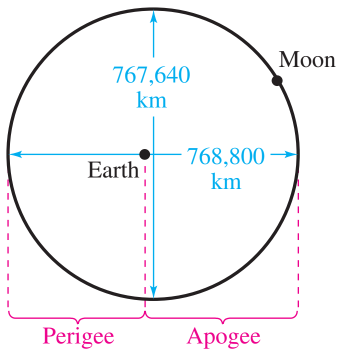


Figure 9.24

> **STUDY TIP**
>
> Note in Example 5 and Figure 9.24 that Earth *is not* the center of the moon's orbit.

Solution

Because $2a = 768{,}800$ and $2b = 767{,}640$, you have

$$a = 384{,}400 \qquad \text{and} \qquad b = 383{,}820$$

which implies that

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ &= \sqrt{384{,}400^2 - 383{,}820^2} \\ &\approx 21{,}108. \end{aligned}$$

So, the greatest distance between the center of Earth and the center of the moon is

$$\begin{aligned} a + c &\approx 384{,}400 + 21{,}108 \\ &= 405{,}508 \text{ kilometers} \end{aligned}$$

and the smallest distance is

$$\begin{aligned} a - c &\approx 384{,}400 - 21{,}108 \\ &= 363{,}292 \text{ kilometers.} \end{aligned}$$

CHECKPOINT Now try Exercise 53.

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

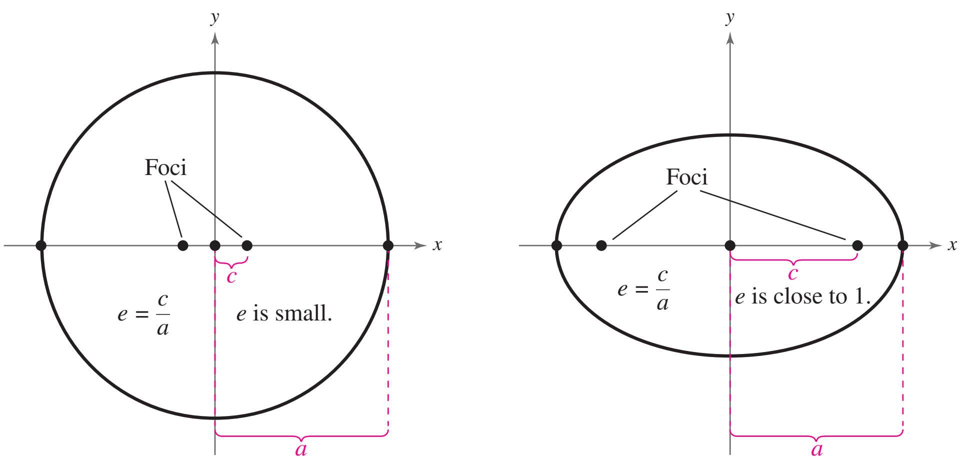
> **Definition of Eccentricity**
>
> The **eccentricity** e of an ellipse is given by the ratio $e = \dfrac{c}{a}$.

Note that $0 < e < 1$ for *every* ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small [see Figure 9.25(a)]. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1 [see Figure 9.25(b)].



(a) (b)

Figure 9.25

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the eight planetary orbits are as follows.

Mercury:	$e \approx 0.2056$	Jupiter:	$e \approx 0.0484$
Venus:	$e \approx 0.0068$	Saturn:	$e \approx 0.0542$
Earth:	$e \approx 0.0167$	Uranus:	$e \approx 0.0472$
Mars:	$e \approx 0.0934$	Neptune:	$e \approx 0.0086$

9.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

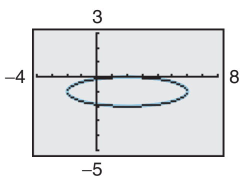
1. An _______ is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points is constant.

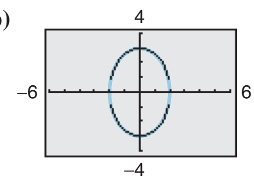
2. The chord joining the vertices of an ellipse is called the _______ , and its midpoint is the _______ of the ellipse.

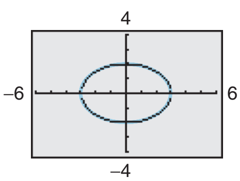
3. The chord perpendicular to the major axis at the center of an ellipse is called the _______ of the ellipse.

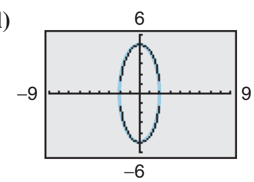
4. You can use the concept of _______ to measure the ovalness of an ellipse.

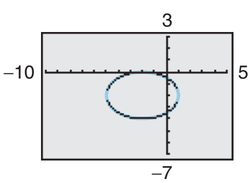
In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

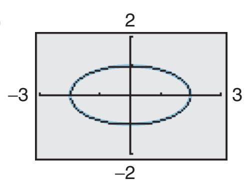
(a)


(b)


(c)


(d)


(e)


(f)


1. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

3. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

4. $\dfrac{x^2}{4} + y^2 = 1$

5. $\dfrac{(x-2)^2}{16} + (y+1)^2 = 1$

6. $\dfrac{(x+2)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

In Exercises 7–12, find the center, vertices, foci, and eccentricity of the ellipse, and sketch its graph. Use a graphing utility to verify your graph.

7. $\dfrac{x^2}{64} + \dfrac{y^2}{9} = 1$

8. $\dfrac{x^2}{16} + \dfrac{y^2}{81} = 1$

9. $\dfrac{(x-4)^2}{16} + \dfrac{(y+1)^2}{25} = 1$

10. $\dfrac{(x+3)^2}{12} + \dfrac{(y-2)^2}{16} = 1$

11. $\dfrac{(x+5)^2}{\frac{9}{4}} + (y-1)^2 = 1$

12. $(x+2)^2 + \dfrac{(y+4)^2}{\frac{1}{4}} = 1$

In Exercises 13–22, (a) find the standard form of the equation of the ellipse, (b) find the center, vertices, foci, and eccentricity of the ellipse, (c) sketch the ellipse, and use a graphing utility to verify your graph.

13. $x^2 + 9y^2 = 36$

14. $16x^2 + y^2 = 16$

15. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

16. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$

17. $6x^2 + 2y^2 + 18x - 10y + 2 = 0$

18. $x^2 + 4y^2 - 6x + 20y - 2 = 0$

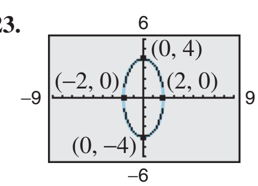
19. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

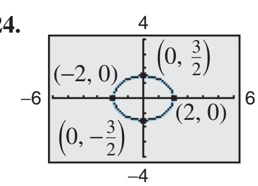
20. $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

21. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

22. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

In Exercises 23–30, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.

23.


24.


25. Vertices: $(\pm 3, 0)$; foci: $(\pm 2, 0)$

26. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$

27. Foci: $(\pm 3, 0)$; major axis of length 8

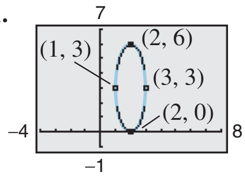
28. Foci: $(\pm 2, 0)$; major axis of length 12

29. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

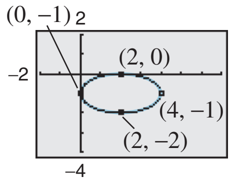
30. Vertical major axis; passes through points $(0, 4)$ and $(2, 0)$

In Exercises 31–40, find the standard form of the equation of the ellipse with the given characteristics.

31.



32.



33. Vertices: (0, 2), (8, 2); minor axis of length 2

34. Foci: (0, 0), (4, 0); major axis of length 6

35. Foci: (0, 0), (0, 8); major axis of length 36

36. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2

37. Vertices: (3, 1), (3, 9); minor axis of length 6

38. Center: (3, 2); $a = 3c$; foci: (1, 2), (5, 2)

39. Center: (0, 4); $a = 2c$; vertices: $(-4, 4)$, (4, 4)

40. Vertices: (5, 0), (5, 12); endpoints of the minor axis: (0, 6), (10, 6)

In Exercises 41–44, find the eccentricity of the ellipse.

41. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

42. $\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$

43. $x^2 + 9y^2 - 10x + 36y + 52 = 0$

44. $4x^2 + 3y^2 - 8x + 18y + 19 = 0$

45. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{4}{5}$.

46. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

47. ***Architecture*** A semielliptical arch over a tunnel for a road through a mountain has a major axis of 100 feet and a height at the center of 40 feet.

(a) Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.

(b) Find an equation of the semielliptical arch over the tunnel.

(c) Determine the height of the arch 5 feet from the edge of the tunnel.

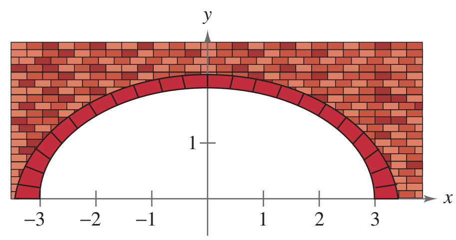
48. ***Architecture*** A semielliptical arch through a railroad underpass has a major axis of 32 feet and a height at the center of 12 feet.

(a) Draw a rectangular coordinate system on a sketch of the underpass with the center of the road entering the underpass at the origin. Identify the coordinates of the known points.

(b) Find an equation of the semielliptical arch over the underpass.

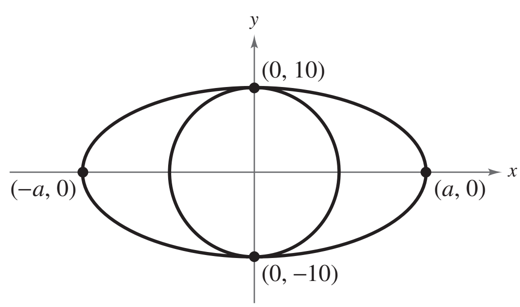
(c) Will a truck that is 10 feet wide and 9 feet tall be able to drive through the underpass without crossing the center line? Explain your reasoning.

49. ***Architecture*** A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base (see figure). The contractor draws the outline of the ellipse on the wall by the method discussed on page 671. Give the required positions of the tacks and the length of the string.



50. ***Statuary Hall*** Statuary Hall is an elliptical room in the United States Capitol Building in Washington, D.C. The room is also referred to as the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. Given that the dimensions of Statuary Hall are 46 feet wide by 97 feet long, find an equation for the shape of the floor surface of the hall. Determine the distance between the foci.

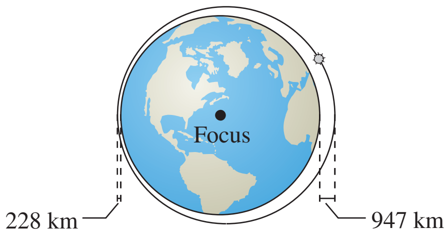
51. ***Geometry*** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis? (*Hint:* The area of an ellipse is given by $A = \pi ab$.)



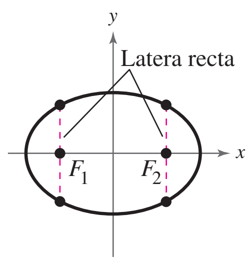
52. ***Astronomy*** Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 0.97. The length of the major axis of the orbit is about 35.88 astronomical units. (An astronomical unit is about 93 million miles.) Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

53. ***Astronomy*** The comet Encke has an elliptical orbit with the sun at one focus. Encke's orbit ranges from 0.34 to 4.08 astronomical units from the sun. Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

54. ***Satellite Orbit*** The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 947 kilometers, and its lowest point was 228 kilometers. The center of Earth was a focus of the elliptical orbit, and the radius of Earth is 6378 kilometers (see figure). Find the eccentricity of the orbit.



55. ***Geometry*** A line segment through a focus with endpoints on an ellipse, perpendicular to the major axis, is called a **latus rectum** of the ellipse. Therefore, an ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because this information yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.



In Exercises 56–59, sketch the ellipse using the latera recta (see Exercise 55).

56. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

57. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

58. $9x^2 + 4y^2 = 36$

59. $5x^2 + 3y^2 = 15$

60. ***Writing*** Write an equation of an ellipse in standard form and graph it on paper. Do not write the equation on your graph. Exchange graphs with another student. Use the graph you receive to reconstruct the equation of the ellipse it represents and find its eccentricity. Compare your results and write a short paragraph discussing your findings.

Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. It is easier to distinguish the graph of an ellipse from the graph of a circle if the eccentricity of the ellipse is large (close to 1).

62. The area of a circle with diameter $d = 2r = 8$ is greater than the area of an ellipse with major axis $2a = 8$.

63. ***Think About It*** At the beginning of this section it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil (see Figure 9.16). If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil is an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

64. ***Exploration*** Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a + b = 20.$$

(a) The area of the ellipse is given by $A = \pi ab$. Write the area of the ellipse as a function of a.

(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table using your equation from part (a) and make a conjecture about the shape of the ellipse with a maximum area.

a	8	9	10	11	12	13
A						

(d) Use a graphing utility to graph the area function to support your conjecture in part (c).

65. ***Think About It*** Find the equation of an ellipse such that for any point on the ellipse, the sum of the distances from the point $(2, 2)$ and $(10, 2)$ is 36.

66. ***Proof*** Show that $a^2 = b^2 + c^2$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > 0, b > 0$, and the distance from the center of the ellipse $(0, 0)$ to a focus is c.

Skills Review

In Exercises 67–70, determine whether the sequence is arithmetic, geometric, or neither.

67. 66, 55, 44, 33, 22, . . .

68. 80, 40, 20, 10, 5, . . .

69. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$

70. $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

In Exercises 71–74, find the sum.

71. $\displaystyle\sum_{n=0}^{6} 3^n$

72. $\displaystyle\sum_{n=0}^{6} (-3)^n$

73. $\displaystyle\sum_{n=1}^{10} 4\left(\frac{3}{4}\right)^{n-1}$

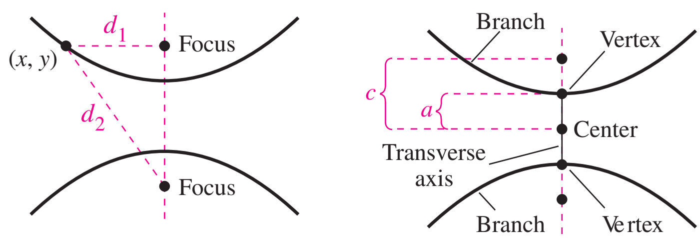
74. $\displaystyle\sum_{n=0}^{10} 5\left(\frac{4}{3}\right)^n$

9.3 Hyperbolas

Introduction

The definition of a **hyperbola** is similar to that of an ellipse. The difference is that for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant; whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

> **Definition of a Hyperbola**
>
> A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, the **foci,** is a positive constant. [See Figure 9.26(a).]



$d_2 - d_1$ is a positive constant.

(a) (b)

Figure 9.26

The graph of a hyperbola has two disconnected parts called the **branches.** The line through the two foci intersects the hyperbola at two points called the **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola [see Figure 9.26(b)]. The development of the **standard form of the equation of a hyperbola** is similar to that of an ellipse. Note that a, b, and c are related differently for hyperbolas than for ellipses. For a hyperbola, the distance between the foci and the center is greater than the distance between the vertices and the center.

> **Standard Equation of a Hyperbola**
>
> The **standard form of the equation of a hyperbola** with center at (h, k) is
>
> $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$
>
> $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$
>
> The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.
>
> $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$
>
> $$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

What you should learn

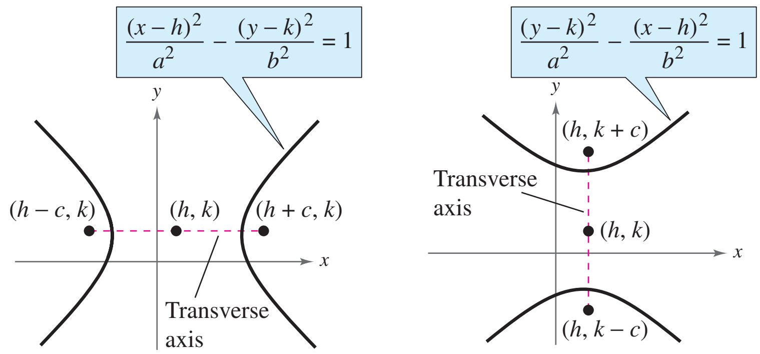
- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.

Why you should learn it

Hyperbolas can be used to model and solve many types of real-life problems. For instance, in Exercise 44 on page 688, hyperbolas are used to locate the position of an explosion that was recorded by three listening stations.

James Foote/Photo Researchers, Inc.

Figure 9.27 shows both the horizontal and vertical orientations for a hyperbola.



Transverse axis is horizontal. ***Transverse axis is vertical.***

Figure 9.27

Example 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci $(-1, 2)$ and $(5, 2)$ and vertices $(0, 2)$ and $(4, 2)$.

Solution

By the Midpoint Formula, the center of the hyperbola occurs at the point $(2, 2)$. Furthermore, $c = 3$ and $a = 2$, and it follows that

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{3^2 - 2^2} \\ &= \sqrt{9 - 4} \\ &= \sqrt{5}. \end{aligned}$$

So, the hyperbola has a horizontal transverse axis and the standard form of the equation of the hyperbola is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{\left(\sqrt{5}\right)^2} = 1.$$

Figure 9.28 shows the hyperbola.

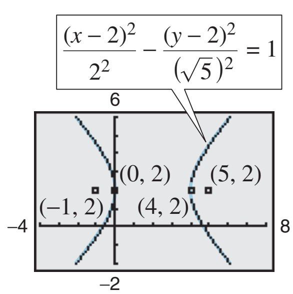


Figure 9.28

CHECKPOINT Now try Exercise 33.

TECHNOLOGY TIP

When using a graphing utility to graph an equation, you must solve the equation for y before entering it into the graphing utility. When graphing equations of conics, it can be difficult to solve for y, which is why it is very important to know the algebra used to solve equations for y.

Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola. The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k), as shown in Figure 9.29.

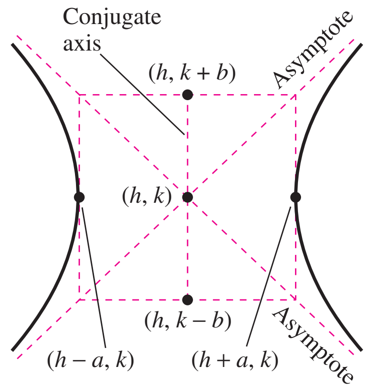


Figure 9.29

> **Asymptotes of a Hyperbola**
>
> $$y = k \pm \frac{b}{a}(x - h)$$ Asymptotes for horizontal transverse axis
>
> $$y = k \pm \frac{a}{b}(x - h)$$ Asymptotes for vertical transverse axis

The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ if the transverse axis is horizontal, and the line segment of length $2b$ joining $(h + b, k)$ and $(h - b, k)$ if the transverse axis is vertical.

Example 2 Sketching a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

Algebraic Solution

$$4x^2 - y^2 = 16 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16} \qquad \text{Divide each side by 16.}$$

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal. So, the vertices occur at $(-2, 0)$ and $(2, 0)$, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure 9.30. Finally, by drawing the asymptotes, $y = 2x$ and $y = -2x$, through the corners of this rectangle, you can complete the sketch, as shown in Figure 9.31.

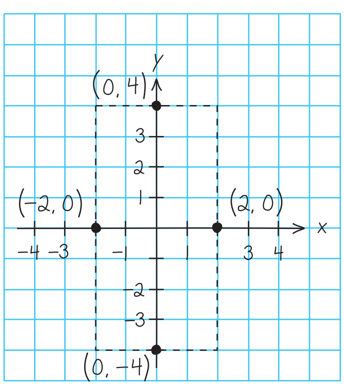


Figure 9.30

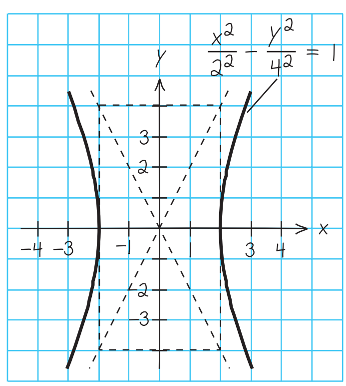


Figure 9.31

CHECKPOINT Now try Exercise 15.

Graphical Solution

Solve the equation of the hyperbola for y as follows.

$$4x^2 - y^2 = 16$$

$$4x^2 - 16 = y^2$$

$$\pm\sqrt{4x^2 - 16} = y$$

Then use a graphing utility to graph $y_1 = \sqrt{4x^2 - 16}$ and $y_2 = -\sqrt{4x^2 - 16}$ in the same viewing window. Be sure to use a square setting. From the graph in Figure 9.32, you can see that the transverse axis is horizontal. You can use the *zoom* and *trace* features to approximate the vertices to be $(-2, 0)$ and $(2, 0)$.

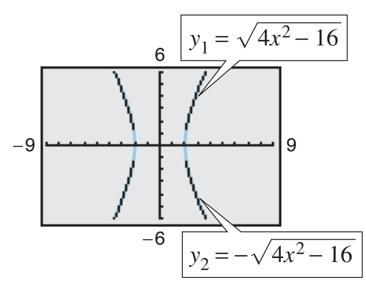


Figure 9.32

Example 3 Finding the Asymptotes of a Hyperbola

Sketch the hyperbola given by

$$4x^2 - 3y^2 + 8x + 16 = 0$$

and find the equations of its asymptotes.

Solution

$$4x^2 - 3y^2 + 8x + 16 = 0 \qquad \text{Write original equation.}$$

$$4(x^2 + 2x) - 3y^2 = -16 \qquad \text{Subtract 16 from each side and factor.}$$

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4(1) \qquad \text{Complete the square.}$$

$$4(x + 1)^2 - 3y^2 = -12 \qquad \text{Write in completed square form.}$$

$$\frac{y^2}{2^2} - \frac{(x + 1)^2}{\left(\sqrt{3}\right)^2} = 1 \qquad \text{Write in standard form.}$$

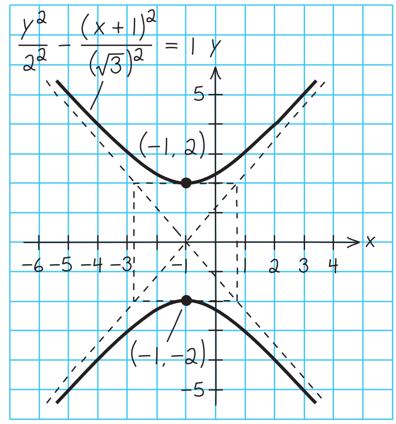


Figure 9.33

From this equation you can conclude that the hyperbola has a vertical transverse axis, is centered at $(-1, 0)$, has vertices $(-1, 2)$ and $(-1, -2)$, and has a conjugate axis with endpoints $\left(-1 - \sqrt{3}, 0\right)$ and $\left(-1 + \sqrt{3}, 0\right)$. To sketch the hyperbola, draw a rectangle through these four points. The asymptotes are the lines passing through the corners of the rectangle, as shown in Figure 9.33. Finally, using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \qquad \text{and} \qquad y = -\frac{2}{\sqrt{3}}(x + 1).$$

CHECKPOINT Now try Exercise 19.

TECHNOLOGY TIP You can use a graphing utility to graph a hyperbola by graphing the upper and lower portions in the same viewing window. For instance, to graph the hyperbola in Example 3, first solve for y to obtain

$$y_1 = 2\sqrt{1 + \frac{(x + 1)^2}{3}} \quad \text{and} \quad y_2 = -2\sqrt{1 + \frac{(x + 1)^2}{3}}.$$

Use a viewing window in which $-9 \le x \le 9$ and $-6 \le y \le 6$. You should obtain the graph shown in Figure 9.34. Notice that the graphing utility does not draw the asymptotes. However, if you trace along the branches, you will see that the values of the hyperbola approach the asymptotes.

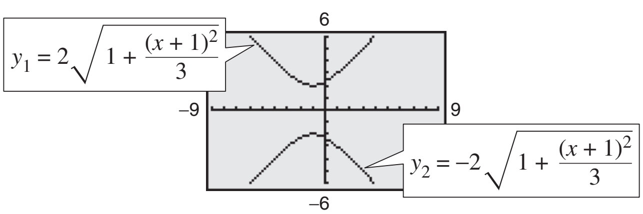


Figure 9.34

Example 4 Using Asymptotes to Find the Standard Equation

Find the standard form of the equation of the hyperbola having vertices $(3, -5)$ and $(3, 1)$ and having asymptotes

$$y = 2x - 8 \qquad \text{and} \qquad y = -2x + 4$$

as shown in Figure 9.35.

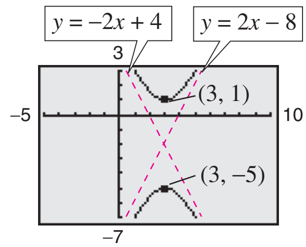


Figure 9.35

Solution

By the Midpoint Formula, the center of the hyperbola is $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. From the original equations, you can determine the slopes of the asymptotes to be

$$m_1 = 2 = \frac{a}{b} \qquad \text{and} \qquad m_2 = -2 = -\frac{a}{b}$$

and because $a = 3$, you can conclude that $b = \frac{3}{2}$. So, the standard form of the equation is

$$\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{\left(\frac{3}{2}\right)^2} = 1.$$

CHECKPOINT Now try Exercise 39.

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a} \qquad \text{Eccentricity}$$

and because $c > a$ it follows that $e > 1$. If the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 9.36(a). If the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 9.36(b).

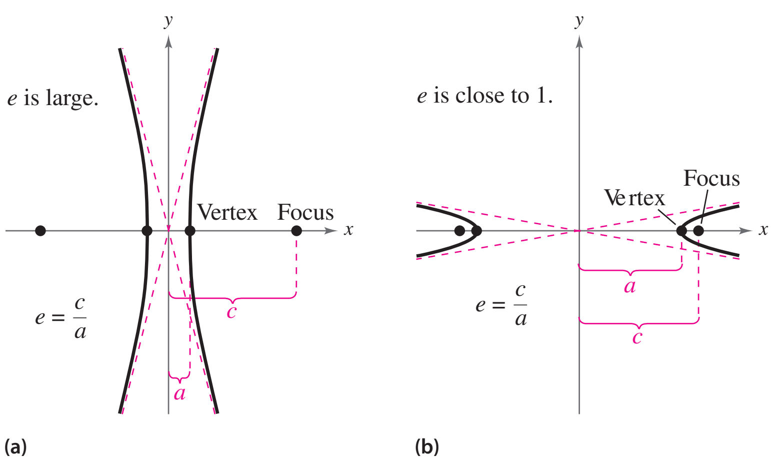


Figure 9.36

Applications

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

Example 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur?

Solution

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 9.37. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{5280}{2} = 2640 \quad \text{and} \quad a = \frac{2200}{2} = 1100.$$

So, $b^2 = c^2 - a^2 = 2640^2 - 1100^2 = 5{,}759{,}600$, and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

CHECKPOINT Now try Exercise 43.

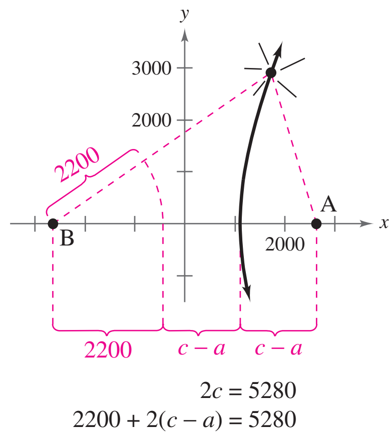


Figure 9.37

Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 9.38. Undoubtedly, there are many comets with parabolic or hyperbolic orbits that have not been identified. You get to see such comets only *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

If p is the distance between the vertex and the focus in meters, and v is the velocity of the comet at the vertex in meters per second, then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of these equations, $M \approx 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

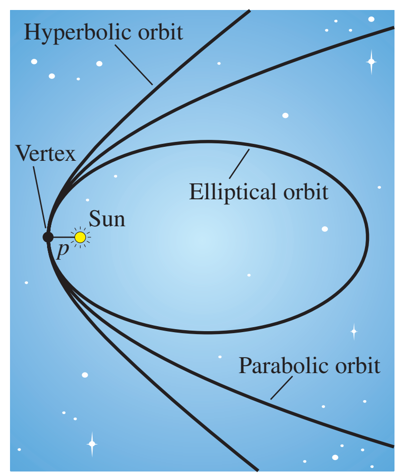


Figure 9.38

General Equations of Conics

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. Circle: $A = C$ $A \neq 0$
2. Parabola: $AC = 0$ $A = 0$ or $C = 0$, but not both.
3. Ellipse: $AC > 0$ A and C have like signs.
4. Hyperbola: $AC < 0$ A and C have unlike signs.

The test above is valid *if* the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, whose graphs are not conics.

Example 6 Classifying Conics from General Equations

Classify the graph of each equation.

a. $4x^2 - 9x + y - 5 = 0$
b. $4x^2 - y^2 + 8x - 6y + 4 = 0$
c. $2x^2 + 4y^2 - 4x + 12y = 0$
d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Solution

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$$AC = 4(0) = 0. \qquad \text{Parabola}$$

So, the graph is a parabola.

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$$AC = 4(-1) < 0. \qquad \text{Hyperbola}$$

So, the graph is a hyperbola.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$$AC = 2(4) > 0. \qquad \text{Ellipse}$$

So, the graph is an ellipse.

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$$A = C = 2. \qquad \text{Circle}$$

So, the graph is a circle.

CHECKPOINT Now try Exercise 49.

STUDY TIP

Notice in Example 6(a) that there is no y^2-term in the equation. Therefore, $C = 0$.

9.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

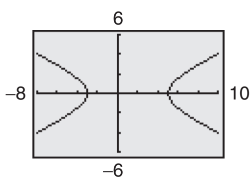
Vocabulary Check

Fill in the blanks.

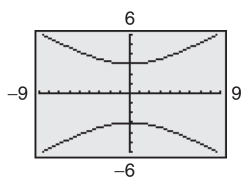
1. A _______ is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points is a positive constant.
2. The graph of a hyperbola has two disconnected parts called _______ .
3. The line segment connecting the vertices of a hyperbola is called the _______ , and the midpoint of the line segment is the _______ of the hyperbola.
4. Each hyperbola has two _______ that intersect at the center of the hyperbola.
5. The general form of the equation of a conic is given by _______ .

In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

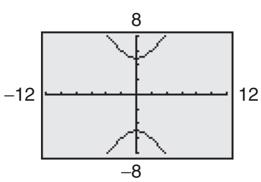
(a)



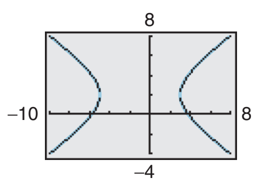
(b)



(c)



(d)



1. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$
2. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$
3. $\dfrac{(x-1)^2}{16} - \dfrac{y^2}{4} = 1$
4. $\dfrac{(x+1)^2}{16} - \dfrac{(y-2)^2}{9} = 1$

In Exercises 5–14, find the center, vertices, foci, and asymptotes of the hyperbola, and sketch its graph using the asymptotes as an aid. Use a graphing utility to verify your graph.

5. $x^2 - y^2 = 1$
6. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$
7. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$
8. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$
9. $\dfrac{y^2}{25} - \dfrac{x^2}{81} = 1$
10. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$
11. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$
12. $\dfrac{(x+3)^2}{144} - \dfrac{(y-2)^2}{25} = 1$
13. $\dfrac{(y+5)^2}{\frac{1}{9}} - \dfrac{(x-1)^2}{\frac{1}{4}} = 1$
14. $\dfrac{(y-1)^2}{\frac{1}{4}} - \dfrac{(x+3)^2}{\frac{1}{16}} = 1$

In Exercises 15–24, (a) find the standard form of the equation of the hyperbola, (b) find the center, vertices, foci, and asymptotes of the hyperbola, (c) sketch the hyperbola, and use a graphing utility to verify your graph.

15. $4x^2 - 9y^2 = 36$
16. $25x^2 - 4y^2 = 100$
17. $2x^2 - 3y^2 = 6$
18. $6y^2 - 3x^2 = 18$
19. $9x^2 - y^2 - 36x - 6y + 18 = 0$
20. $x^2 - 9y^2 + 36y - 72 = 0$
21. $x^2 - 9y^2 + 2x - 54y - 80 = 0$
22. $16y^2 - x^2 + 2x + 64y + 63 = 0$
23. $9y^2 - x^2 + 2x + 54y + 62 = 0$
24. $9x^2 - y^2 + 54x + 10y + 55 = 0$

In Exercises 25–30, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.

25. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$
26. Vertices: $(\pm 3, 0)$; foci: $(\pm 6, 0)$
27. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 5x$
28. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$
29. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$
30. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

In Exercises 31–42, find the standard form of the equation of the hyperbola with the given characteristics.

31. Vertices: $(2, 0)$, $(6, 0)$; foci: $(0, 0)$, $(8, 0)$

32. Vertices: $(2, 3)$, $(2, -3)$; foci: $(2, 5)$, $(2, -5)$

33. Vertices: $(4, 1)$, $(4, 9)$; foci: $(4, 0)$, $(4, 10)$

34. Vertices: $(-2, 1)$, $(2, 1)$; foci: $(-3, 1)$, $(3, 1)$

35. Vertices: $(2, 3)$, $(2, -3)$;
passes through the point $(0, 5)$

36. Vertices: $(-2, 1)$, $(2, 1)$;
passes through the point $(5, 4)$

37. Vertices: $(0, 4)$, $(0, 0)$;
passes through the point $(\sqrt{5}, -1)$

38. Vertices: $(1, 2)$, $(1, -2)$;
passes through the point $(0, \sqrt{5})$

39. Vertices: $(1, 2)$, $(3, 2)$;
asymptotes: $y = x$, $y = 4 - x$

40. Vertices: $(3, 0)$, $(3, -6)$;
asymptotes: $y = x - 6$, $y = -x$

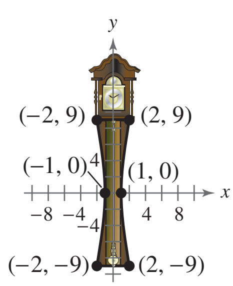
41. Vertices: $(0, 2)$, $(6, 2)$;
asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

42. Vertices: $(3, 0)$, $(3, 4)$;
asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

43. ***Sound Location*** You and a friend live 4 miles apart (on the same "east-west" street) and are talking on the phone. You hear a clap of thunder from lightning in a storm, and 18 seconds later your friend hears the thunder. Find an equation that gives the possible places where the lightning could have occurred. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

44. ***Sound Location*** Three listening stations located at $(3300, 0)$, $(3300, 1100)$, and $(-3300, 0)$ monitor an explosion. The last two stations detect the explosion 1 second and 4 seconds after the first, respectively. Determine the coordinates of the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

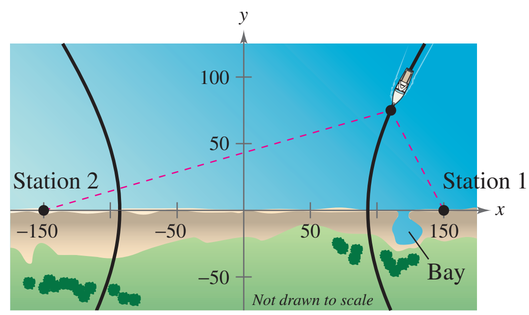
45. ***Pendulum*** The base for a pendulum of a clock has the shape of a hyperbola (see figure).



(a) Write an equation of the cross section of the base.

(b) Each unit in the coordinate plane represents $\frac{1}{2}$ foot. Find the width of the base of the pendulum 4 inches from the bottom.

46. ***Navigation*** Long distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at coordinates $(-150, 0)$ and $(150, 0)$, and that a ship is traveling on a hyperbolic path with coordinates $(x, 75)$ (see figure).



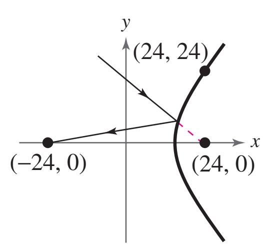
(a) Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

(b) Determine the distance between the ship and station 1 when the ship reaches the shore.

(c) The captain of the ship wants to enter a bay located between the two stations. The bay is 30 miles from station 1. What should be the time difference between the pulses?

(d) The ship is 60 miles offshore when the time difference in part (c) is obtained. What is the position of the ship?

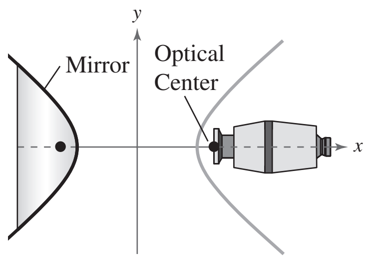
47. ***Hyperbolic Mirror*** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at a focus will be reflected to the other focus. The focus of a hyperbolic mirror (see figure) has coordinates $(24, 0)$. Find the vertex of the mirror if the mount at the top edge of the mirror has coordinates $(24, 24)$.



48. ***Panoramic Photo*** A panoramic photo can be taken using a hyperbolic mirror. The camera is pointed toward the vertex of the mirror and the camera's optical center is positioned at one focus of the mirror (see figure). An equation for the cross-section of the mirror is

$$\frac{y^2}{25} - \frac{x^2}{16} = 1.$$

Find the distance from the camera's optical center to the mirror.



In Exercises 49–58, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

49. $9x^2 + 4y^2 - 18x + 16y - 119 = 0$

50. $x^2 + y^2 - 4x - 6y - 23 = 0$

51. $16x^2 - 9y^2 + 32x + 54y - 209 = 0$

52. $x^2 + 4x - 8y + 20 = 0$

53. $y^2 + 12x + 4y + 28 = 0$

54. $4x^2 + 25y^2 + 16x + 250y + 541 = 0$

55. $x^2 + y^2 + 2x - 6y = 0$

56. $y^2 - x^2 + 2x - 6y - 8 = 0$

57. $x^2 - 6x - 2y + 7 = 0$

58. $9x^2 + 4y^2 - 90x + 8y + 228 = 0$

Synthesis

True or False? **In Exercises 59–62, determine whether the statement is true or false. Justify your answer.**

59. In the standard form of the equation of a hyperbola, the larger the ratio of b to a, the larger the eccentricity of the hyperbola.

60. In the standard form of the equation of a hyperbola, the trivial solution of two intersecting lines occurs when $b = 0$.

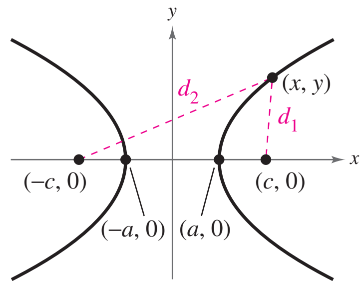
61. If $D \neq 0$ and $E \neq 0$, then the graph of $x^2 - y^2 + Dx + Ey = 0$ is a hyperbola.

62. If the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, where $a, b > 0$, intersect at right angles, then $a = b$.

63. ***Think About It*** Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.

64. ***Writing*** Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.

65. Use the figure to show that $|d_2 - d_1| = 2a$.



66. ***Think About It*** Find the equation of the hyperbola for any point on which, the difference between its distances from the points (2, 2) and (10, 2) is 6.

67. ***Proof*** Show that $c^2 = a^2 + b^2$ for the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where the distance from the center of the hyperbola (0, 0) to a focus is c.

68. ***Proof*** Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	*Condition*
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

Skills Review

In Exercises 69–72, perform the indicated operation.

69. Subtract: $(x^3 - 3x^2) - (6 - 2x - 4x^2)$

70. Multiply: $\left(3x - \frac{1}{2}\right)(x + 4)$

71. Divide: $\dfrac{x^3 - 3x + 4}{x + 2}$

72. Expand: $[(x + y) + 3]^2$

In Exercises 73–78, factor the polynomial completely.

73. $x^3 - 16x$

74. $x^2 + 14x + 49$

75. $2x^3 - 24x^2 + 72x$

76. $6x^3 - 11x^2 - 10x$

77. $16x^3 + 54$

78. $4 - x + 4x^2 - x^3$

9.4 Rotation and Systems of Quadratic Equations

What you should learn

- Rotate the coordinate axes to eliminate the *xy*-term in equations of conics.
- Use the discriminant to classify conics.
- Solve systems of quadratic equations.

Why you should learn it

As illustrated in Exercises 3–14 on page 697, rotation of the coordinate axes can help you identify the graph of a general second-degree equation.

Rotation

In the preceding section, you learned that the equation of a conic with axes parallel to the coordinate axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \qquad \text{Horizontal or vertical axes}$$

In this section, you will study the equations of conics whose axes are rotated so that they are not parallel to either the x-axis or the y-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy\text{-plane}$$

To eliminate this xy-term, you can use a procedure called **rotation of axes.** The objective is to rotate the x- and y-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure 9.39.

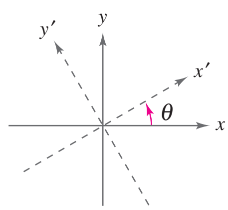


Figure 9.39

After the rotation, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \qquad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no xy-term, you can obtain a standard form by completing the square. The following theorem identifies how much to rotate the axes to eliminate the xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

Rotation of Axes to Eliminate an *xy*-Term (See the proof on page 738.)

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where $\cot 2\theta = \dfrac{A - C}{B}$.

The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

Example 1 Rotation of Axes for a Hyperbola

Rotate the axes to eliminate the xy-term in the equation $xy - 1 = 0$. Then write the equation in standard form and sketch its graph.

Solution

Because $A = 0$, $B = 1$, and $C = 0$, you have

$$\cot 2\theta = \frac{A - C}{B} = 0 \quad \Longrightarrow \quad 2\theta = \frac{\pi}{2} \quad \Longrightarrow \quad \theta = \frac{\pi}{4}$$

which implies that

$$\begin{aligned} x &= x' \cos \frac{\pi}{4} - y' \sin \frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin \frac{\pi}{4} + y' \cos \frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' + y'}{\sqrt{2}}. \end{aligned}$$

The equation in the $x'y'$-system is obtained by substituting these expressions into the equation $xy - 1 = 0$.

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1 \qquad \text{Write in standard form.}$$

In the $x'y'$-system, this is a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$, as shown in Figure 9.40. To find the coordinates of the vertices in the xy-system, substitute the coordinates $(\pm\sqrt{2}, 0)$ into the equations

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

This substitution yields the vertices $(1, 1)$ and $(-1, -1)$ in the xy-system. Note also that the asymptotes of the hyperbola have equations $y' = \pm x'$, which correspond to the original x- and y-axes.

CHECKPOINT Now try Exercise 3.

STUDY TIP

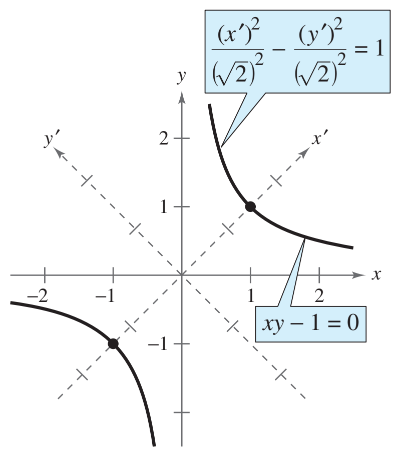
Remember that the substitutions

$$x = x' \cos \theta - y' \sin \theta$$

and

$$y = x' \sin \theta + y' \cos \theta$$

were developed to eliminate the $x'y'$-term in the rotated system. You can use this as a check on your work. In other words, if your final equation contains an $x'y'$-term, you know that you made a mistake.



Vertices:
In $x'y'$-system: $(\sqrt{2}, 0), (-\sqrt{2}, 0)$
In xy-system: $(1, 1), (-1, -1)$
Figure 9.40

Example 2 Rotation of Axes for an Ellipse

Rotate the axes to eliminate the xy-term in the equation

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

Then write the equation in standard form and sketch its graph.

Prerequisite Skills

To review conics, see Sections 9.1–9.3.

Solution

Because $A = 7$, $B = -6\sqrt{3}$, and $C = 13$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

which implies that $\theta = \pi/6$. The equation in the $x'y'$-system is obtained by making the substitutions

$$x = x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6}$$
$$= x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{3}x' - y'}{2}$$

and

$$y = x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6}$$
$$= x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{x' + \sqrt{3}y'}{2}$$

into the original equation. So, you have

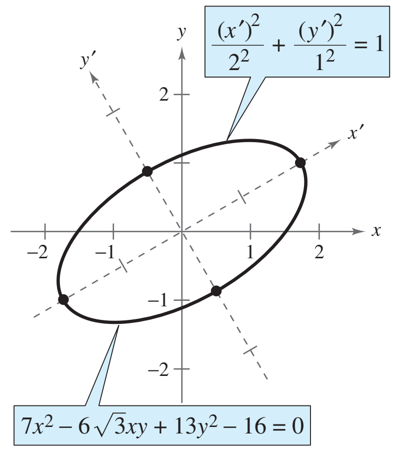
$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$

which simplifies to

$$4(x')^2 + 16(y')^2 - 16 = 0$$
$$4(x')^2 + 16(y')^2 = 16$$
$$\frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} = 1. \qquad \text{Write in standard form.}$$

This is the equation of an ellipse centered at the origin with vertices $(\pm 2, 0)$ in the $x'y'$-system, as shown in Figure 9.41.



Vertices:

In x′y′-system: $(\pm 2, 0)$, $(0, \pm 1)$

In xy-system: $(\sqrt{3}, 1)$, $(-\sqrt{3}, 1)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Figure 9.41

CHECKPOINT Now try Exercise 11.

Example 3 Rotation of Axes for a Parabola

Rotate the axes to eliminate the xy-term in the equation

$$x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0.$$

Then write the equation in standard form and sketch its graph.

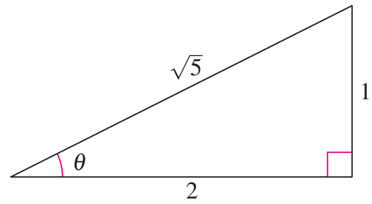


Figure 9.42

Solution

Because $A = 1$, $B = -4$, and $C = 4$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

Using the identity $\cot 2\theta = (\cot^2 \theta - 1)/(2 \cot \theta)$ produces

$$\cot 2\theta = \frac{3}{4} = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

from which you obtain the equation

$$4 \cot^2 \theta - 4 = 6 \cot \theta$$

$$4 \cot^2 \theta - 6 \cot \theta - 4 = 0$$

$$(2 \cot \theta - 4)(2 \cot \theta + 1) = 0.$$

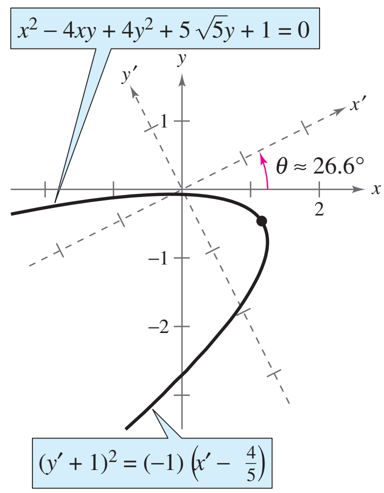
Considering $0 < \theta < \pi/2$, you have $2 \cot \theta = 4$. So,

$$\cot \theta = 2 \quad \Longrightarrow \quad \theta \approx 26.6°.$$

From the triangle in Figure 9.42, you obtain $\sin \theta = 1/\sqrt{5}$ and $\cos \theta = 2/\sqrt{5}$. So, you use the substitutions

$$x = x' \cos \theta - y' \sin \theta = x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta = x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) = \frac{x' + 2y'}{\sqrt{5}}.$$



Vertex:

In x′y′-system: $\left(\frac{4}{5}, -1\right)$

In xy-system: $\left(\frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}}\right)$

Figure 9.43

Substituting these expressions into the original equation, you have

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 + 5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$

which simplifies as follows.

$$5(y')^2 + 5x' + 10y' + 1 = 0$$

$$5[(y')^2 + 2y'] = -5x' - 1 \qquad \text{Group terms.}$$

$$5(y' + 1)^2 = -5x' + 4 \qquad \text{Write in completed square form.}$$

$$(y' + 1)^2 = (-1)\left(x' - \tfrac{4}{5}\right) \qquad \text{Write in standard form.}$$

The graph of this equation is a parabola with vertex at $\left(\frac{4}{5}, -1\right)$ in the $x'y'$-plane. Its axis is parallel to the x'-axis in the $x'y'$-system, as shown in Figure 9.43.

CHECKPOINT Now try Exercise 13.

Invariants Under Rotation

In the rotation of axes theorem listed at the beginning of this section, note that the constant term is the same in both equations—that is, $F' = F$. Such quantities are **invariant under rotation.** The next theorem lists some other rotation invariants.

Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$
2. $A + C = A' + C'$
3. $B^2 - 4AC = (B')^2 - 4A'C'$

You can use the results of this theorem to classify the graph of a second-degree equation *with* an xy-term in much the same way that you classify the graph of a second-degree equation *without* an xy-term. Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C'. \quad \text{Discriminant}$$

This quantity is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Now, from the classification procedure given in Section 9.3, you know that the sign of $A'C'$ determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Consequently, the sign of $B^2 - 4AC$ will determine the type of graph for the original equation, as shown in the following classification.

Classification of Conics by the Discriminant

The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, except in degenerate cases, determined by its discriminant as follows.

1. Ellipse or circle: $B^2 - 4AC < 0$
2. Parabola: $B^2 - 4AC = 0$
3. Hyperbola: $B^2 - 4AC > 0$

For example, in the general equation $3x^2 + 7xy + 5y^2 - 6x - 7y + 15 = 0$, you have $A = 3$, $B = 7$, and $C = 5$. So, the discriminant is

$$B^2 - 4AC = 7^2 - 4(3)(5) = 49 - 60 = -11.$$

Because $-11 < 0$, the graph of the equation is an ellipse or a circle.

Example 4 Rotations and Graphing Utilities

For each equation, classify the graph of the equation, use the Quadratic Formula to solve for y, and then use a graphing utility to graph the equation.

a. $2x^2 - 3xy + 2y^2 - 2x = 0$ **b.** $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

c. $3x^2 + 8xy + 4y^2 - 7 = 0$

Solution

a. Because $B^2 - 4AC = 9 - 16 < 0$, the graph is a circle or an ellipse. Solve for y as follows.

$$2x^2 - 3xy + 2y^2 - 2x = 0 \qquad \text{Write original equation.}$$

$$2y^2 - 3xy + (2x^2 - 2x) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-(-3x) \pm \sqrt{(-3x)^2 - 4(2)(2x^2 - 2x)}}{2(2)}$$

$$y = \frac{3x \pm \sqrt{x(16 - 7x)}}{4}$$

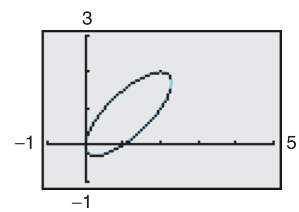


Figure 9.44

Graph both of the equations to obtain the ellipse shown in Figure 9.44.

$$y_1 = \frac{3x + \sqrt{x(16 - 7x)}}{4} \qquad \text{Top half of ellipse}$$

$$y_2 = \frac{3x - \sqrt{x(16 - 7x)}}{4} \qquad \text{Bottom half of ellipse}$$

b. Because $B^2 - 4AC = 36 - 36 = 0$, the graph is a parabola.

$$x^2 - 6xy + 9y^2 - 2y + 1 = 0 \qquad \text{Write original equation.}$$

$$9y^2 - (6x + 2)y + (x^2 + 1) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{(6x + 2) \pm \sqrt{(6x + 2)^2 - 4(9)(x^2 + 1)}}{2(9)}$$

$$y = \frac{3x + 1 \pm \sqrt{2(3x - 4)}}{9}$$

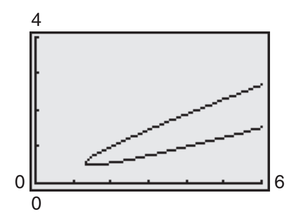


Figure 9.45

Graph both of the equations to obtain the parabola shown in Figure 9.45.

c. Because $B^2 - 4AC = 64 - 48 > 0$, the graph is a hyperbola.

$$3x^2 + 8xy + 4y^2 - 7 = 0 \qquad \text{Write original equation.}$$

$$4y^2 + 8xy + (3x^2 - 7) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-8x \pm \sqrt{(8x)^2 - 4(4)(3x^2 - 7)}}{2(4)}$$

$$y = \frac{-2x \pm \sqrt{x^2 + 7}}{2}$$

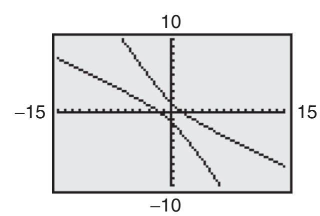


Figure 9.46

Graph both of the equations to obtain the hyperbola shown in Figure 9.46.

CHECKPOINT Now try Exercise 27.

Systems of Quadratic Equations

To find the points of intersection of two conics, you can use elimination or substitution, as demonstrated in Examples 5 and 6.

> **TECHNOLOGY SUPPORT**
>
> For instructions on how to use the *intersect* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Example 5 Solving a Quadratic System by Elimination

Solve the system of quadratic equations $\begin{cases} x^2 + y^2 - 16x + 39 = 0 & \text{Equation 1} \\ x^2 - y^2 - 9 = 0 & \text{Equation 2} \end{cases}$.

Algebraic Solution

You can eliminate the y^2-term by adding the two equations. The resulting equation can then be solved for x.

$$2x^2 - 16x + 30 = 0$$

$$2(x - 3)(x - 5) = 0$$

There are two real solutions: $x = 3$ and $x = 5$. The corresponding y-values are $y = 0$ and $y = \pm 4$. So, the solutions of the system are $(3, 0)$, $(5, 4)$, and $(5, -4)$.

CHECKPOINT Now try Exercise 41.

Graphical Solution

Begin by solving each equation for y as follows.

$$y = \pm\sqrt{-x^2 + 16x - 39} \qquad y = \pm\sqrt{x^2 - 9}$$

Use a graphing utility to graph all four equations $y_1 = \sqrt{-x^2 + 16x - 39}$, $y_2 = -\sqrt{-x^2 + 16x - 39}$, $y_3 = \sqrt{x^2 - 9}$, and $y_4 = -\sqrt{x^2 - 9}$ in the same viewing window. Use the *intersect* feature of the graphing utility to approximate the points of intersection to be $(3, 0)$, $(5, 4)$, and $(5, -4)$, as shown in Figure 9.47.

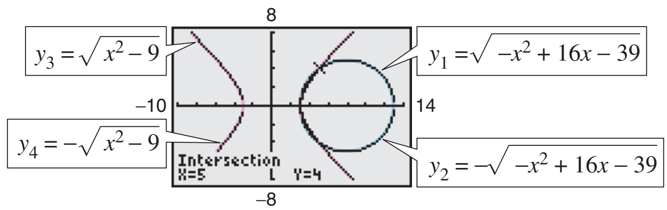


Figure 9.47

Example 6 Solving a Quadratic System by Substitution

Solve the system of quadratic equations $\begin{cases} x^2 + 4y^2 - 4x - 8y + 4 = 0 & \text{Equation 1} \\ x^2 + 4y - 4 = 0 & \text{Equation 2} \end{cases}$.

Solution

Because Equation 2 has no y^2-term, solve the equation for y tobtain $y = 1 - \frac{1}{4}x^2$. Next, substitute this into Equation 1 and solve for x.

$$x^2 + 4\left(1 - \tfrac{1}{4}x^2\right)^2 - 4x - 8\left(1 - \tfrac{1}{4}x^2\right) + 4 = 0$$

$$x^2 + 4 - 2x^2 + \tfrac{1}{4}x^4 - 4x - 8 + 2x^2 + 4 = 0$$

$$x^4 + 4x^2 - 16x = 0$$

$$x(x - 2)(x^2 + 2x + 8) = 0$$

In factored form, you can see that the equation has two real solutions: $x = 0$ and $x = 2$. The corresponding values of y are $y = 1$ and $y = 0$. This implies that the solutions of the system of equations are $(0, 1)$ and $(2, 0)$, as shown in Figure 9.48.

CHECKPOINT Now try Exercise 47.

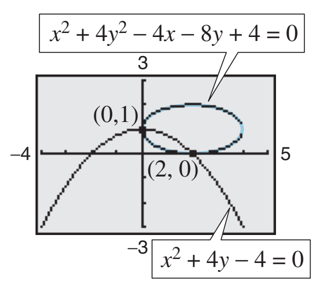


Figure 9.48

9.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The procedure used to eliminate the xy-term in a general second-degree equation is called _______ of _______ .
2. Quantities that are equal in both the original equation of a conic and the equation of the rotated conic are _______ .
3. The quantity $B^2 - 4AC$ is called the _______ of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

In Exercises 1 and 2, the $x'y'$-coordinate system has been rotated θ degrees from the xy-coordinate system. The coordinates of a point in the xy-coordinate system are given. Find the coordinates of the point in the rotated coordinate system.

1. $\theta = 90°, (0, 3)$
2. $\theta = 45°, (3, 3)$

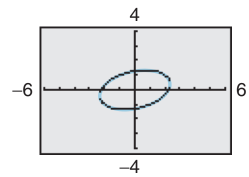
In Exercises 3–14, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.

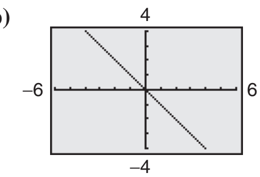
3. $xy + 1 = 0$
4. $xy - 2 = 0$
5. $x^2 - 4xy + y^2 + 1 = 0$
6. $xy + x - 2y + 3 = 0$
7. $xy - 2y - 4x = 0$
8. $2x^2 - 3xy - 2y^2 + 10 = 0$
9. $5x^2 - 6xy + 5y^2 - 12 = 0$
10. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
11. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
12. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
13. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
14. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

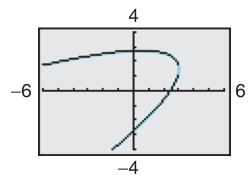
In Exercises 15–20, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.

15. $x^2 + 3xy + y^2 = 20$
16. $x^2 - 4xy + 2y^2 = 8$
17. $17x^2 + 32xy - 7y^2 = 75$
18. $40x^2 + 36xy + 25y^2 = 52$
19. $32x^2 + 48xy + 8y^2 = 50$
20. $4x^2 - 12xy + 9y^2 + (4\sqrt{13} - 12)x - (6\sqrt{13} + 8)y = 91$

In Exercises 21–26, match the graph with its equation. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)


(b)


(c)


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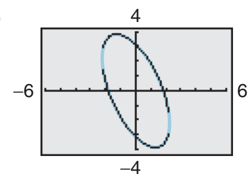

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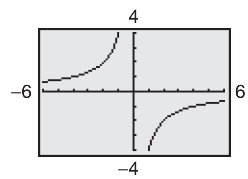
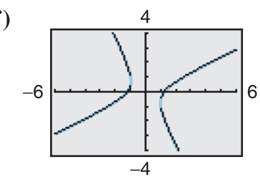


(f)


21. $xy + 4 = 0$
22. $x^2 + 2xy + y^2 = 0$
23. $-2x^2 + 3xy + 2y^2 + 3 = 0$
24. $x^2 - xy + 3y^2 - 5 = 0$
25. $3x^2 + 2xy + y^2 - 10 = 0$
26. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

In Exercises 27–34, (a) use the discriminant to classify the graph of the equation, (b) use the Quadratic Formula to solve for y, and (c) use a graphing utility to graph the equation.

27. $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
28. $x^2 - 4xy - 2y^2 - 6 = 0$
29. $15x^2 - 8xy + 7y^2 - 45 = 0$
30. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
31. $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
32. $36x^2 - 60xy + 25y^2 + 9y = 0$

33. $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$

34. $x^2 + xy + 4y^2 + x + y - 4 = 0$

In Exercises 35–38, sketch (if possible) the graph of the degenerate conic.

35. $y^2 - 16x^2 = 0$

36. $x^2 + y^2 - 2x + 6y + 10 = 0$

37. $x^2 + 2xy + y^2 - 1 = 0$

38. $x^2 - 10xy + y^2 = 0$

In Exercises 39–46, solve the system of quadratic equations algebraically by the method of elimination. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

39. $\begin{cases} x^2 + y^2 - 4 = 0 \\ 3x - y^2 = 0 \end{cases}$

40. $\begin{cases} 4x^2 + 9y^2 - 36y = 0 \\ x^2 + y^2 - 27 = 0 \end{cases}$

41. $\begin{cases} -4x^2 - y^2 - 16x + 24y - 16 = 0 \\ 4x^2 + y^2 + 40x - 24y + 208 = 0 \end{cases}$

42. $\begin{cases} x^2 - 4y^2 - 20x - 64y - 172 = 0 \\ 16x^2 + 4y^2 - 320x + 64y + 1600 = 0 \end{cases}$

43. $\begin{cases} x^2 - y^2 - 12x + 16y - 64 = 0 \\ x^2 + y^2 - 12x - 16y + 64 = 0 \end{cases}$

44. $\begin{cases} x^2 + 4y^2 - 2x - 8y + 1 = 0 \\ -x^2 + 2x - 4y - 1 = 0 \end{cases}$

45. $\begin{cases} -16x^2 - y^2 + 24y - 80 = 0 \\ 16x^2 + 25y^2 - 400 = 0 \end{cases}$

46. $\begin{cases} 16x^2 - y^2 + 16y - 128 = 0 \\ y^2 - 48x - 16y - 32 = 0 \end{cases}$

In Exercises 47–52, solve the system of quadratic equations algebraically by the method of substitution. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

47. $\begin{cases} 2x^2 - y^2 + 6 = 0 \\ 2x + y = 0 \end{cases}$

48. $\begin{cases} 6x^2 + 3y^2 - 12 = 0 \\ x + y - 2 = 0 \end{cases}$

49. $\begin{cases} 10x^2 - 25y^2 - 100x + 160 = 0 \\ y^2 - 2x + 16 = 0 \end{cases}$

50. $\begin{cases} 4x^2 - y^2 - 8x + 6y - 9 = 0 \\ 2x^2 - 3y^2 + 4x + 18y - 43 = 0 \end{cases}$

51. $\begin{cases} xy + x - 2y + 3 = 0 \\ x^2 + 4y^2 - 9 = 0 \end{cases}$

52. $\begin{cases} 5x^2 - 2xy + 5y^2 - 12 = 0 \\ x + y - 1 = 0 \end{cases}$

Synthesis

***True or False?* In Exercises 53 and 54, determine whether the statement is true or false. Justify your answer.**

53. The graph of $x^2 + xy + ky^2 + 6x + 10 = 0$, where k is any constant less than $\frac{1}{4}$, is a hyperbola.

54. After using a rotation of axes to eliminate the xy-term from an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the coefficients of the x^2- and y^2-terms remain A and C, respectively.

Skills Review

In Exercises 55–58, sketch the graph of the rational function. Identify all intercepts and asymptotes.

55. $g(x) = \dfrac{2}{2 - x}$

56. $f(x) = \dfrac{2x}{2 - x}$

57. $h(t) = \dfrac{t^2}{2 - t}$

58. $g(s) = \dfrac{2}{4 - s^2}$

In Exercises 59–62, if possible, find (a) *AB*, (b) *BA*, and (c) A^2.

59. $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 6 \\ 5 & -1 \end{bmatrix}$

60. $A = \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ -3 & 8 \end{bmatrix}$

61. $A = \begin{bmatrix} 4 & -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

62. $A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 5 \\ 3 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -3 \\ -4 & 5 & -1 \\ 6 & 3 & 2 \end{bmatrix}$

In Exercises 63–70, graph the function.

63. $f(x) = |x + 3|$

64. $f(x) = |x - 4| + 1$

65. $g(x) = \sqrt{4 - x^2}$

66. $g(x) = \sqrt{3x - 2}$

67. $h(t) = -(t - 2)^3 + 3$

68. $h(t) = \frac{1}{2}(t + 4)^3$

69. $f(t) = [\![t - 5]\!] + 1$

70. $f(t) = -2[\![t]\!] + 3$

In Exercises 71–74, find the area of the triangle.

71. $C = 110°, a = 8, b = 12$

72. $B = 70°, a = 25, c = 16$

73. $a = 11, b = 18, c = 10$

74. $a = 23, b = 35, c = 27$

9.5 Parametric Equations

Plane Curves

Up to this point, you have been representing a graph by a single equation involving *two* variables such as x and y. In this section, you will study situations in which it is useful to introduce a *third* variable to represent a curve in the plane.

To see the usefulness of this procedure, consider the path of an object that is propelled into the air at an angle of 45°. If the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

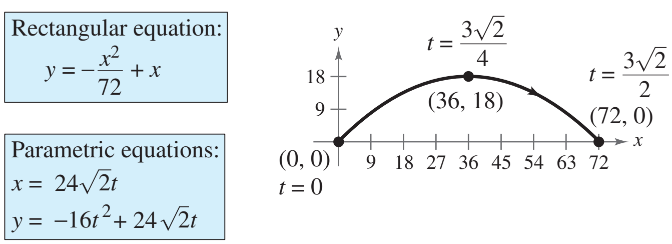
$$y = -\frac{x^2}{72} + x \qquad \text{Rectangular equation}$$

as shown in Figure 9.49. However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point (x, y) on the path. To determine this time, you can introduce a third variable t, called a **parameter.** It is possible to write both x and y as functions of t to obtain the **parametric equations**

$$x = 24\sqrt{2}t \qquad \text{Parametric equation for } x$$

$$y = -16t^2 + 24\sqrt{2}t. \qquad \text{Parametric equation for } y$$

From this set of equations you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$, and so on.



Curvilinear motion: two variables for position, one variable for time
Figure 9.49

For this particular motion problem, x and y are continuous functions of t, and the resulting path is a **plane curve.** (Recall that a *continuous function* is one whose graph can be traced without lifting the pencil from the paper.)

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a **plane curve** C. The equations given by

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

are **parametric equations** for C, and t is the **parameter.**

What you should learn

- Evaluate sets of parametric equations for given values of the parameter.
- Graph curves that are represented by sets of parametric equations.
- Rewrite sets of parametric equations as single rectangular equations by eliminating the parameter.
- Find sets of parametric equations for graphs.

Why you should learn it

Parametric equations are useful for modeling the path of an object. For instance, in Exercise 57 on page 706, a set of parametric equations is used to model the path of a football.

Sketching a Plane Curve

One way to sketch a curve represented by a pair of parametric equations is to plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in the order of *increasing* values of t, you trace the curve in a specific direction. This is called the **orientation** of the curve.

Example 1 Sketching a Plane Curve

Sketch the curve given by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \qquad -2 \le t \le 3.$$

Describe the orientation of the curve.

Solution

Using values of t in the interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

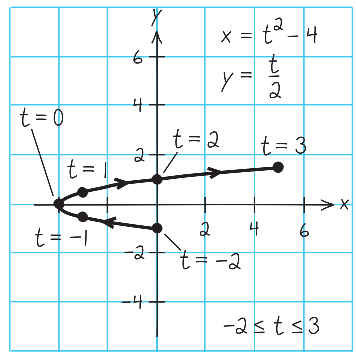


Figure 9.50

By plotting these points in the order of increasing t, you obtain the curve shown in Figure 9.50. The arrows on the curve indicate its orientation as t increases from -2 to 3. So, if a particle were moving on this curve, it would start at $(0, -1)$ and then move along the curve to the point $\left(5, \frac{3}{2}\right)$.

CHECKPOINT Now try Exercises 7(a) and (b).

Note that the graph shown in Figure 9.50 does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

Two different sets of parametric equations can have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \qquad -1 \le t \le \tfrac{3}{2}$$

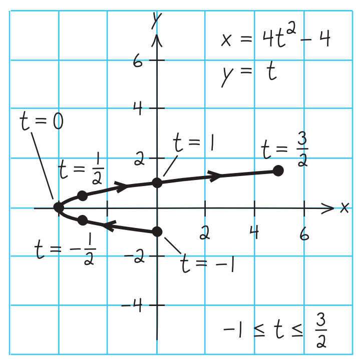


Figure 9.51

has the same graph as the set given in Example 1. However, by comparing the values of t in Figures 9.50 and 9.51, you can see that this second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

TECHNOLOGY TIP Most graphing utilities have a *parametric* mode. So, another way to display a curve represented by a pair of parametric equations is to use a graphing utility, as shown in Example 2. For instructions on how to use the *parametric* mode, see Appendix A; for specifice keystrokes, go to this textbook's *Online Study Center*.

Example 2 Using a Graphing Utility in Parametric Mode

Use a graphing utility to graph the curves represented by the parametric equations. Using the graph and the Vertical Line Test, for which curve is y a function of x?

a. $x = t^2,\ y = t^3$ **b.** $x = t,\ y = t^3$ **c.** $x = t^2,\ y = t$

Solution

Begin by setting the graphing utility to *parametric* mode. When choosing a viewing window, you must set not only minimum and maximum values of x and y, but also minimum and maximum values of t.

a. Enter the parametric equations for x and y, as shown in Figure 9.52. Use the viewing window shown in Figure 9.53. The curve is shown in Figure 9.54. From the graph, you can see that y *is not* a function of x.

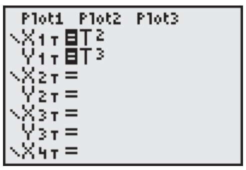


Figure 9.52

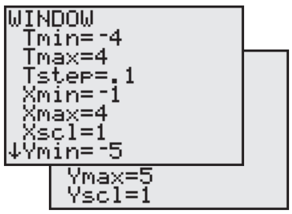


Figure 9.53

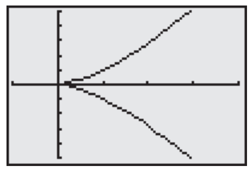


Figure 9.54

b. Enter the parametric equations for x and y, as shown in Figure 9.55. Use the viewing window shown in Figure 9.56. The curve is shown in Figure 9.57. From the graph, you can see that y *is* a function of x.

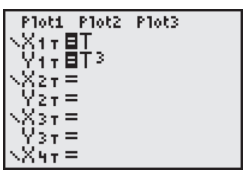


Figure 9.55

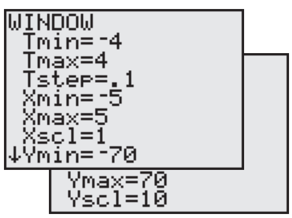


Figure 9.56

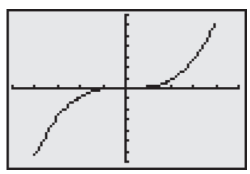


Figure 9.57

c. Enter the parametric equations for x and y, as shown in Figure 9.58. Use the viewing window shown in Figure 9.59. The curve is shown in Figure 9.60. From the graph, you can see that y *is not* a function of x.

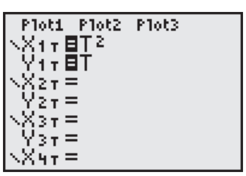


Figure 9.58

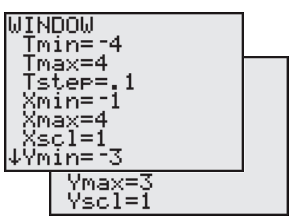


Figure 9.59

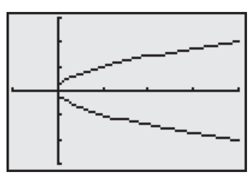


Figure 9.60

CHECKPOINT Now try Exercise 7(c).

Prerequisite Skills

See Section 1.3 to review the Vertical Line Test.

Exploration

Use a graphing utility set in *parametric* mode to graph the curve

$$x = t \quad \text{and} \quad y = 1 - t^2$$

Set the viewing window so that $-4 \le x \le 4$ and $-12 \le y \le 2$. Now, graph the curve with various settings for t. Use the following.

a. $0 \le t \le 3$

b. $-3 \le t \le 0$

c. $-3 \le t \le 3$

Compare the curves given by the different t settings. Repeat this experiment using $x = -t$. How does this change the results?

TECHNOLOGY TIP

Notice in Example 2 that in order to set the viewing windows of parametric graphs, you have to scroll down to enter the Ymax and Yscl values.

Eliminating the Parameter

Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular equations (in x and y). The process of finding the rectangular equation is called **eliminating the parameter.**

Parametric equations	→	Solve for t in one equation.	→	Substitute in second equation.	→	Rectangular equation
$x = t^2 - 4$ $y = \frac{1}{2}t$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Now you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$.

When converting equations from parametric to rectangular form, you may need to alter the domain of the rectangular equation so that its graph matches the graph of the parametric equations. This situation is demonstrated in Example 3.

Example 3 Eliminating the Parameter

Identify the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}.$$

Solution

Solving for t in the equation for x produces

$$x^2 = \frac{1}{t+1} \quad \text{or} \quad \frac{1}{x^2} = t + 1$$

which implies that $t = (1/x^2) - 1$. Substituting in the equation for y, you obtain the rectangular equation

$$y = \frac{t}{t+1}$$

$$= \frac{\left(\frac{1}{x^2}\right) - 1}{\left(\frac{1}{x^2}\right) - 1 + 1} = \frac{\frac{1-x^2}{x^2}}{\left(\frac{1}{x^2}\right)} \cdot \frac{x^2}{x^2} = 1 - x^2.$$

From the rectangular equation, you can recognize that the curve is a parabola that opens downward and has its vertex at $(0, 1)$, as shown in Figure 9.61. The rectangular equation is defined for all values of x. The parametric equation for x, however, is defined only when $t > -1$. From the graph of the parametric equations, you can see that x is always positive, as shown in Figure 9.62. So, you should restrict the domain of x to positive values, as shown in Figure 9.63.

CHECKPOINT Now try Exercise 7(d).

STUDY TIP

It is important to realize that eliminating the parameter is primarily an aid to curve sketching. If the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to determine the *position*, *direction*, and *speed* at a given time.

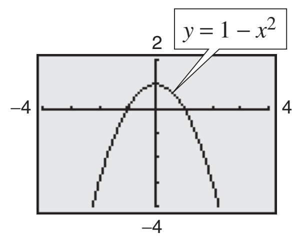


Figure 9.61

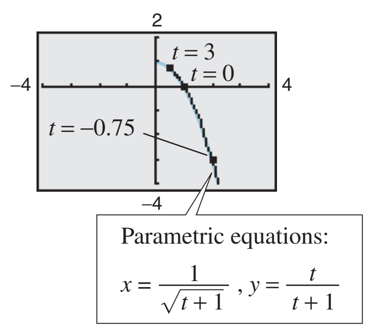


Figure 9.62

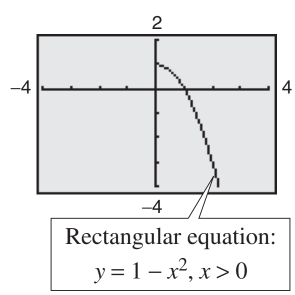


Figure 9.63

Example 4 Eliminating the Parameter

Sketch the curve represented by $x = 3\cos\theta$ and $y = 4\sin\theta$, $0 \le \theta \le 2\pi$, by eliminating the parameter.

Solution

Begin by solving for $\cos\theta$ and $\sin\theta$ in the equations.

$$\cos\theta = \frac{x}{3} \quad \text{and} \quad \sin\theta = \frac{y}{4} \qquad \text{Solve for } \cos\theta \text{ and } \sin\theta.$$

Use the identity $\sin^2\theta + \cos^2\theta = 1$ to form an equation involving only x and y.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Pythagorean identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \text{Substitute } \frac{x}{3} \text{ for } \cos\theta \text{ and } \frac{y}{4} \text{ for } \sin\theta.$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{Rectangular equation}$$

From this rectangular equation, you can see that the graph is an ellipse centered at $(0, 0)$, with vertices $(0, 4)$ and $(0, -4)$, and minor axis of length $2b = 6$, as shown in Figure 9.64. Note that the elliptic curve is traced out *counterclockwise*.

CHECKPOINT Now try Exercise 23.

Exploration

In Example 4, you make use of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to sketch an ellipse. Which trigonometric identity would you use to obtain the graph of a hyperbola? Sketch the curve represented by $x = 3\sec\theta$ and $y = 4\tan\theta$, $0 \le \theta \le 2\pi$, by eliminating the parameter.

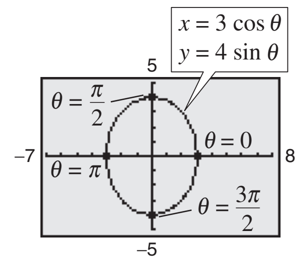


Figure 9.64

Finding Parametric Equations for a Graph

How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is further demonstrated in Example 5.

Example 5 Finding Parametric Equations for a Given Graph

Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the parameters (a) $t = x$ and (b) $t = 1 - x$.

Solution

a. Letting $t = x$, you obtain the following parametric equations.

$$x = t \qquad \text{Parametric equation for } x$$

$$y = 1 - t^2 \qquad \text{Parametric equation for } y$$

The graph of these equations is shown in Figure 9.65.

b. Letting $t = 1 - x$, you obtain the following parametric equations.

$$x = 1 - t \qquad \text{Parametric equation for } x$$

$$y = 1 - (1 - t)^2 = 2t - t^2 \qquad \text{Parametric equation for } y$$

The graph of these equations is shown in Figure 9.66. Note that the graphs in Figures 9.65 and 9.66 have opposite orientations.

CHECKPOINT Now try Exercise 45.

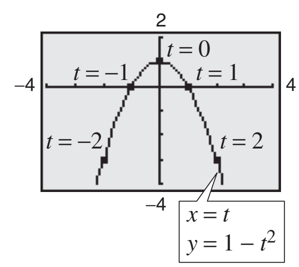


Figure 9.65

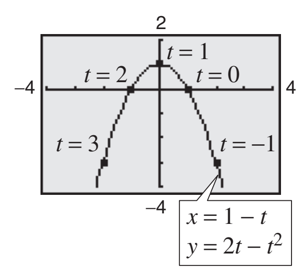


Figure 9.66

9.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

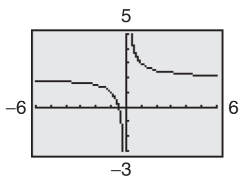
1. If f and g are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a _______ C. The equations given by $x = f(t)$ and $y = g(t)$ are _______ for C, and t is the _______ .

2. The _______ of a curve is the direction in which the curve is traced out for increasing values of the parameter.

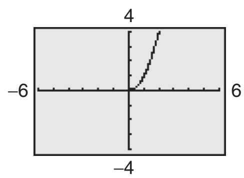
3. The process of converting a set of parametric equations to rectangular form is called _______ the _______ .

In Exercises 1–6, match the set of parametric equations with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

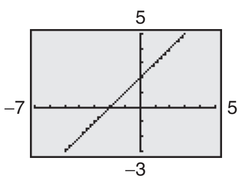
(a)



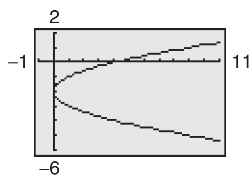
(b)



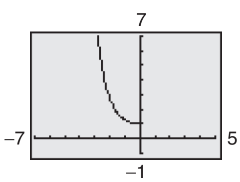
(c)



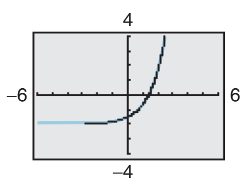
(d)



(e)



(f)



1. $x = t,\ y = t + 2$

2. $x = t^2,\ y = t - 2$

3. $x = \sqrt{t},\ y = t$

4. $x = \dfrac{1}{t},\ y = t + 2$

5. $x = \ln t,\ y = \frac{1}{2}t - 2$

6. $x = -2\sqrt{t},\ y = e^t$

7. Consider the parametric equations $x = \sqrt{t}$ and $y = 2 - t$.

(a) Create a table of x- and y-values using $t = 0, 1, 2, 3,$ and 4.

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How does the graph differ from those in parts (b) and (c)?

8. Consider the parametric equations $x = 4\cos^2\theta$ and $y = 4\sin\theta$.

(a) Create a table of x- and y-values using $\theta = -\pi/2, -\pi/4, 0, \pi/4,$ and $\pi/2$.

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How does the graph differ from those in parts (b) and (c)?

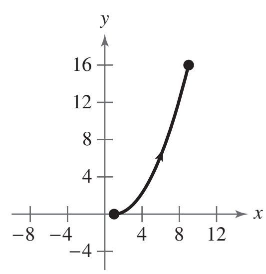
Library of Parent Functions **In Exercises 9 and 10, determine the plane curve whose graph is shown.**

9. (a) $x = t^2$
$y = 2t + 1$

(b) $x = 2t + 1$
$y = t^2$

(c) $x = 2t - 1$
$y = t^2$

(d) $x = -2t + 1$
$y = t^2$

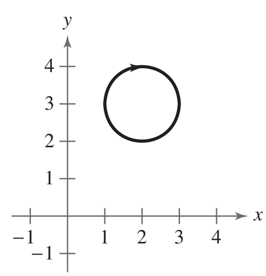


10. (a) $x = 2 - \cos\theta$
$y = 3 - \sin\theta$

(b) $x = 3 - \cos\theta$
$y = 2 + \sin\theta$

(c) $x = 2 + \cos\theta$
$y = 3 - \sin\theta$

(d) $x = 3 + \cos\theta$
$y = 2 - \sin\theta$



In Exercises 11–26, sketch the curve represented by the parametric equations (indicate the orientation of the curve). Use a graphing utility to confirm your result. Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.

11. $x = t, y = -4t$

12. $x = t, y = \frac{1}{2}t$

13. $x = 3t - 3, y = 2t + 1$

14. $x = 3 - 2t, y = 2 + 3t$

15. $x = \frac{1}{4}t, y = t^2$

16. $x = t, y = t^3$

17. $x = t + 2, y = t^2$

18. $x = \sqrt{t}, y = 1 - t$

19. $x = 2t, y = |t - 2|$

20. $x = |t - 1|, y = t + 2$

21. $x = 2 \cos \theta, y = 3 \sin \theta$

22. $x = \cos \theta, y = 4 \sin \theta$

23. $x = e^{-t}, y = e^{3t}$

24. $x = e^{2t}, y = e^t$

25. $x = t^3, y = 3 \ln t$

26. $x = \ln 2t, y = 2t^2$

In Exercises 27–32, use a graphing utility to graph the curve represented by the parametric equations.

27. $x = 4 + 3 \cos \theta$
$y = -2 + \sin \theta$

28. $x = 4 + 3 \cos \theta$
$y = -2 + 2 \sin \theta$

29. $x = 4 \sec \theta$
$y = 2 \tan \theta$

30. $x = \sec \theta$
$y = \tan \theta$

31. $x = t/2$
$y = \ln(t^2 + 1)$

32. $x = 10 - 0.01e^t$
$y = 0.4t^2$

In Exercises 33 and 34, determine how the plane curves differ from each other.

33. (a) $x = t$
$y = 2t + 1$
(b) $x = \cos \theta$
$y = 2 \cos \theta + 1$
(c) $x = e^{-t}$
$y = 2e^{-t} + 1$
(d) $x = e^t$
$y = 2e^t + 1$

34. (a) $x = 2\sqrt{t}$
$y = 4 - \sqrt{t}$
(b) $x = 2\sqrt[3]{t}$
$y = 4 - \sqrt[3]{t}$
(c) $x = 2(t + 1)$
$y = 3 - t$
(d) $x = -2t^2$
$y = 4 + t^2$

In Exercises 35–38, eliminate the parameter and obtain the standard form of the rectangular equation.

35. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1)$
$y = y_1 + t(y_2 - y_1)$

36. Circle: $x = h + r \cos \theta, y = k + r \sin \theta$

37. Ellipse: $x = h + a \cos \theta, y = k + b \sin \theta$

38. Hyperbola: $x = h + a \sec \theta, y = k + b \tan \theta$

In Exercises 39–42, use the results of Exercises 37–40 to find a set of parametric equations for the line or conic.

39. Line: passes through $(1, 4)$ and $(6, -3)$

40. Circle: center: $(2, 5)$; radius: 4

41. Ellipse: vertices: $(\pm 5, 0)$; foci: $(\pm 4, 0)$

42. Hyperbola: vertices: $(0, \pm 1)$; foci: $(0, \pm 2)$

In Exercises 43–48, find two different sets of parametric equations for the given rectangular equation. (There are many correct answers.)

43. $y = 5x - 3$

44. $y = 4 - 7x$

45. $y = \dfrac{1}{x}$

46. $y = \dfrac{1}{2x}$

47. $y = 6x^2 - 5$

48. $y = x^3 + 2x$

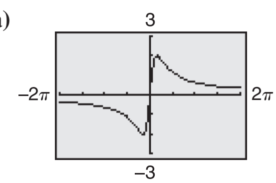
In Exercises 49 and 50, use a graphing utility to graph the curve represented by the parametric equations.

49. Witch of Agnesi: $x = 2 \cot \theta, y = 2 \sin^2 \theta$

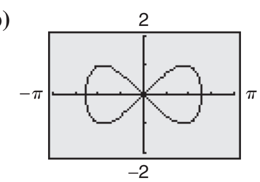
50. Folium of Descartes: $x = \dfrac{3t}{1 + t^3}, y = \dfrac{3t^2}{1 + t^3}$

In Exercises 51–54, match the parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

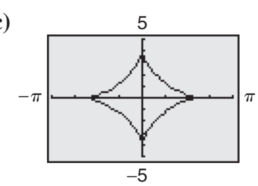
(a)



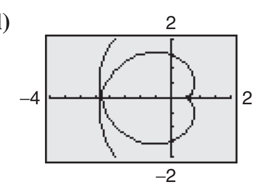
(b)



(c)



(d)



51. Lissajous curve: $x = 2 \cos \theta, y = \sin 2\theta$

52. Evolute of ellipse: $x = 2 \cos^3 \theta, y = 4 \sin^3 \theta$

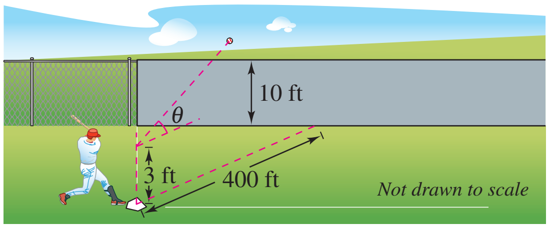
53. Involute of circle: $x = \frac{1}{2}(\cos \theta + \theta \sin \theta)$
$y = \frac{1}{2}(\sin \theta - \theta \cos \theta)$

54. Serpentine curve: $x = \frac{1}{2} \cot \theta, y = 4 \sin \theta \cos \theta$

Projectile Motion **In Exercises 55–58, consider a projectile launched at a height of h feet above the ground at an angle of θ with the horizontal. The initial velocity is v_0 feet per second and the path of the projectile is modeled by the parametric equations**

$$x = (v_0 \cos \theta)t \text{ and } y = h + (v_0 \sin \theta)t - 16t^2.$$

55. The center field fence in a ballpark is 10 feet high and 400 feet from home plate. A baseball is hit at a point 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).



(a) Write a set of parametric equations for the path of the baseball.

(b) Use a graphing utility to graph the path of the baseball for $\theta = 15°$. Is the hit a home run?

(c) Use a graphing utility to graph the path of the baseball for $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle required for the hit to be a home run.

56. The right field fence in a ballpark is 10 feet high and 314 feet from home plate. A baseball is hit at a point 2.5 feet above the ground. It leaves the bat at an angle of $\theta = 40°$ with the horizontal at a speed of 105 feet per second.

(a) Write a set of parametric equations for the path of the baseball.

(b) Use a graphing utility to graph the path of the baseball and approximate its maximum height.

(c) Use a graphing utility to find the horizontal distance that the baseball travels. Is the hit a home run?

(d) Explain how you could find the result in part (c) algebraically.

57. The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal.

(a) Write a set of parametric equations for the path of the football.

(b) Find the speed of the football when it is released.

(c) Use a graphing utility to graph the path of the football and approximate its maximum height.

(d) Find the time the receiver has to position himself after the quarterback releases the football.

58. To begin a football game, a kicker kicks off from his team's 35-yard line. The football is kicked at an angle of 50° with the horizontal at an initial velocity of 85 feet per second.

(a) Write a set of parametric equations for the path of the kick.

(b) Use a graphing utility to graph the path of the kick and approximate its maximum height.

(c) Use a graphing utility to find the horizontal distance that the kick travels.

(d) Explain how you could find the result in part (c) algebraically.

Synthesis

True or False? **In Exercises 59–62, determine whether the statement is true or false. Justify your answer.**

59. The two sets of parametric equations $x = t$, $y = t^2 + 1$ and $x = 3t$, $y = 9t^2 + 1$ correspond to the same rectangular equation.

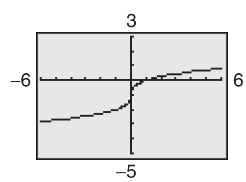
60. Because the graph of the parametric equations $x = t^2$, $y = t^2$ and $x = t$, $y = t$ both represent the line $y = x$, they are the same plane curve.

61. If y is a function of t and x is a function of t, then y must be a function of x.

62. The parametric equations $x = at + h$ and $y = bt + k$, where $a \neq 0$ and $b \neq 0$, represent a circle centered at (h, k), if $a = b$.

63. As θ increases, the ellipse given by the parametric equations $x = \cos \theta$ and $y = 2 \sin \theta$ is traced out *counterclockwise*. Find a parametric representation for which the same ellipse is traced out *clockwise*.

64. ***Think About It*** The graph of the parametric equations $x = t^3$ and $y = t - 1$ is shown below. Would the graph change for the equations $x = (-t)^3$ and $y = -t - 1$? If so, how would it change?



Skills Review

In Exercises 65–68, check for symmetry with respect to both axes and the origin. Then determine whether the function is even, odd, or neither.

65. $f(x) = \dfrac{4x^2}{x^2 + 1}$

66. $f(x) = \sqrt{x}$

67. $y = e^x$

68. $(x - 2)^2 = y + 4$

9.6 Polar Coordinates

What you should learn

- Plot points and find multiple representations of points in the polar coordinate system.
- Convert points from rectangular to polar form and vice versa.
- Convert equations from rectangular to polar form and vice versa.

Why you should learn it

Polar coordinates offer a different mathematical perspective on graphing. For instance, in Exercises 5–12 on page 711, you see that a polar coordinate can be written in more than one way.

Introduction

So far, you have been representing graphs of equations as collections of points (x, y) in the rectangular coordinate system, where x and y represent the directed distances from the coordinate axes to the point (x, y). In this section, you will study a second coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 9.67. Then each point P in the plane can be assigned **polar coordinates** (r, θ) as follows.

1. $r =$ *directed distance* from O to P

2. $\theta =$ *directed angle*, counterclockwise from the polar axis to segment $\overline{OP}$

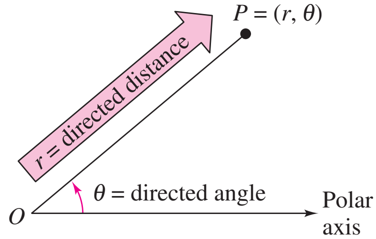


Figure 9.67

Example 1 Plotting Points in the Polar Coordinate System

a. The point $(r, \theta) = (2, \pi/3)$ lies two units from the pole on the terminal side of the angle $\theta = \pi/3$, as shown in Figure 9.68.

b. The point $(r, \theta) = (3, -\pi/6)$ lies three units from the pole on the terminal side of the angle $\theta = -\pi/6$, as shown in Figure 9.69.

c. The point $(r, \theta) = (3, 11\pi/6)$ coincides with the point $(3, -\pi/6)$, as shown in Figure 9.70.

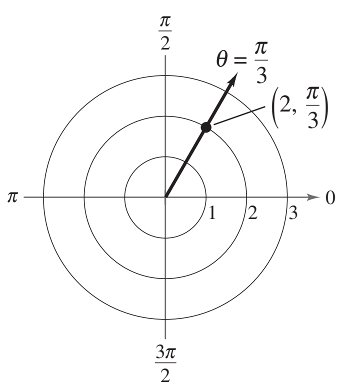


Figure 9.68

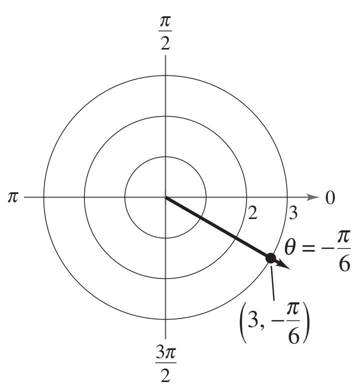


Figure 9.69

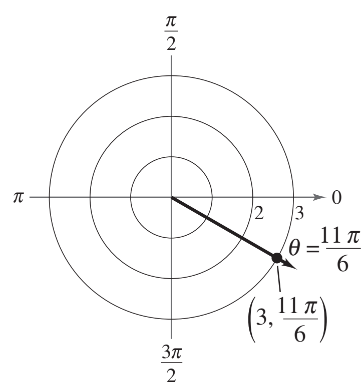


Figure 9.70

CHECKPOINT Now try Exercise 5.

In rectangular coordinates, each point (x, y) has a unique representation. This is not true for polar coordinates. For instance, the coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point, as illustrated in Example 1. Another way to obtain multiple representations of a point is to use negative values for r. Because r is a *directed distance*, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. In general, the point (r, θ) can be represented as

$$(r, \theta) = (r, \theta \pm 2n\pi) \qquad \text{or} \qquad (r, \theta) = (-r, \theta \pm (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Example 2 Multiple Representations of Points

Plot the point $(3, -3\pi/4)$ and find three additional polar representations of this point, using $-2\pi < \theta < 2\pi$.

Solution

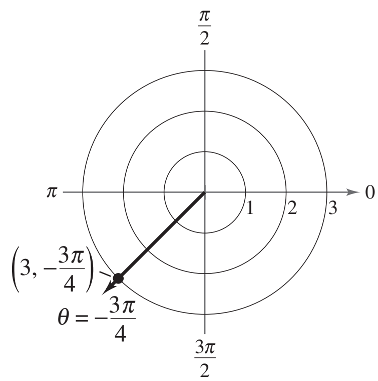
The point is shown in Figure 9.71. Three other representations are as follows.

$$\left(3, -\frac{3\pi}{4} + 2\pi\right) = \left(3, \frac{5\pi}{4}\right) \qquad \text{Add } 2\pi \text{ to } \theta.$$

$$\left(-3, -\frac{3\pi}{4} - \pi\right) = \left(-3, -\frac{7\pi}{4}\right) \qquad \text{Replace } r \text{ by } -r; \text{ subtract } \pi \text{ from } \theta.$$

$$\left(-3, -\frac{3\pi}{4} + \pi\right) = \left(-3, \frac{\pi}{4}\right) \qquad \text{Replace } r \text{ by } -r; \text{ add } \pi \text{ to } \theta.$$

CHECKPOINT Now try Exercise 7.



$\left(3, -\frac{3\pi}{4}\right) = \left(3, \frac{5\pi}{4}\right) = \left(-3, -\frac{7\pi}{4}\right) = \left(-3, \frac{\pi}{4}\right) = \ldots$

Figure 9.71

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 9.72. Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan \theta = \frac{y}{x}, \qquad \cos \theta = \frac{x}{r}, \qquad \text{and} \qquad \sin \theta = \frac{y}{r}.$$

You can show that the same relationships hold for $r < 0$.

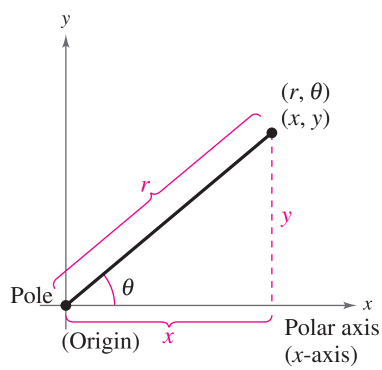


Figure 9.72

Coordinate Conversion

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

$$x = r\cos\theta \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

$$y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2$$

Example 3 Polar-to-Rectangular Conversion

Convert each point to rectangular coordinates.

a. $(2, \pi)$ **b.** $\left(\sqrt{3}, \dfrac{\pi}{6}\right)$

Solution

a. For the point $(r, \theta) = (2, \pi)$, you have the following.

$$x = r\cos\theta = 2\cos\pi = -2$$

$$y = r\sin\theta = 2\sin\pi = 0$$

The rectangular coordinates are $(x, y) = (-2, 0)$. (See Figure 9.73.)

b. For the point $(r, \theta) = \left(\sqrt{3}, \pi/6\right)$, you have the following.

$$x = r\cos\theta = \sqrt{3}\cos\frac{\pi}{6} = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$$

$$y = r\sin\theta = \sqrt{3}\sin\frac{\pi}{6} = \sqrt{3}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

The rectangular coordinates are $(x, y) = \left(3/2, \sqrt{3}/2\right)$. (See Figure 9.73.)

CHECKPOINT Now try Exercise 13.

Example 4 Rectangular-to-Polar Conversion

Convert each point to polar coordinates.

a. $(-1, 1)$ **b.** $(0, 2)$

Solution

a. For the second-quadrant point $(x, y) = (-1, 1)$, you have

$$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$$

$$\theta = \frac{3\pi}{4}.$$

Because θ lies in the same quadrant as (x, y), use positive r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

So, *one* set of polar coordinates is $(r, \theta) = \left(\sqrt{2}, 3\pi/4\right)$, as shown in Figure 9.74.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose

$$\theta = \pi/2 \quad \text{and} \quad r = 2.$$

This implies that one set of polar coordinates is $(r, \theta) = (2, \pi/2)$, as shown in Figure 9.75.

CHECKPOINT Now try Exercise 29.

Exploration

Set your graphing utility to *polar* mode. Then graph the equation $r = 3$. (Use a viewing window in which $0 \le \theta \le 2\pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.) You should obtain a circle of radius 3.

a. Use the *trace* feature to cursor around the circle. Can you locate the point $(3, 5\pi/4)$?

b. Can you locate other representations of the point $(3, 5\pi/4)$? If so, explain how you did it.

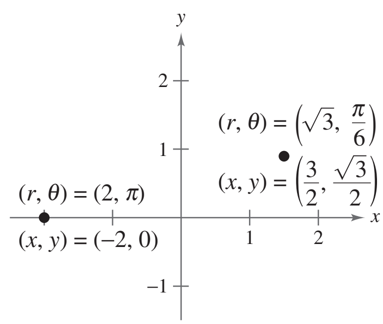


Figure 9.73

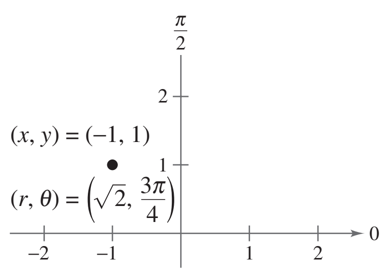


Figure 9.74

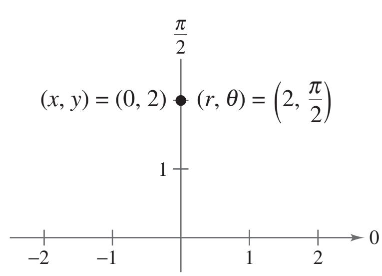


Figure 9.75

Equation Conversion

By comparing Examples 3 and 4, you see that point conversion from the polar to the rectangular system is straightforward, whereas point conversion from the rectangular to the polar system is more involved. For equations, the opposite is true. To convert a rectangular equation to polar form, you simply replace x by $r\cos\theta$ and y by $r\sin\theta$. For instance, the rectangular equation $y = x^2$ can be written in polar form as follows.

$$y = x^2 \qquad \text{Rectangular equation}$$

$$r\sin\theta = (r\cos\theta)^2 \qquad \text{Polar equation}$$

$$r = \sec\theta\tan\theta \qquad \text{Simplest form}$$

On the other hand, converting a polar equation to rectangular form requires considerable ingenuity.

Example 5 demonstrates several polar-to-rectangular conversions that enable you to sketch the graphs of some polar equations.

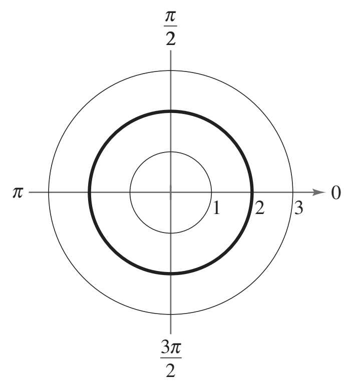


Figure 9.76

Example 5 Converting Polar Equations to Rectangular Form

Describe the graph of each polar equation and find the corresponding rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec\theta$

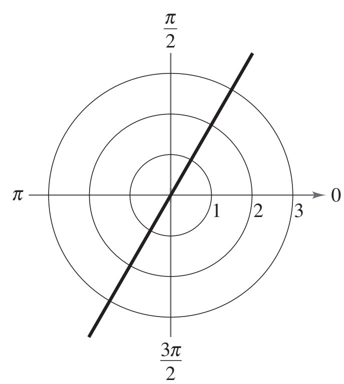


Figure 9.77

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2, as shown in Figure 9.76. You can confirm this by converting to rectangular form, using the relationship $r^2 = x^2 + y^2$.

$$\underbrace{r = 2}_{\text{Polar equation}} \quad\Longrightarrow\quad r^2 = 2^2 \quad\Longrightarrow\quad \underbrace{x^2 + y^2 = 2^2}_{\text{Rectangular equation}}$$

b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis, as shown in Figure 9.77. To convert to rectangular form, you make use of the relationship $\tan\theta = y/x$.

$$\underbrace{\theta = \frac{\pi}{3}}_{\text{Polar equation}} \quad\Longrightarrow\quad \tan\theta = \sqrt{3} \quad\Longrightarrow\quad \underbrace{y = \sqrt{3}x}_{\text{Rectangular equation}}$$

c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you convert to rectangular form by using the relationship $r\cos\theta = x$.

$$\underbrace{r = \sec\theta}_{\text{Polar equation}} \quad\Longrightarrow\quad r\cos\theta = 1 \quad\Longrightarrow\quad \underbrace{x = 1}_{\text{Rectangular equation}}$$

Now you can see that the graph is a vertical line, as shown in Figure 9.78.

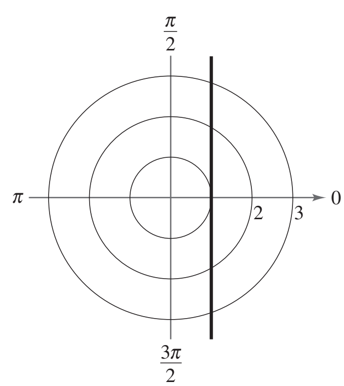


Figure 9.78

CHECKPOINT Now try Exercise 83.

9.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

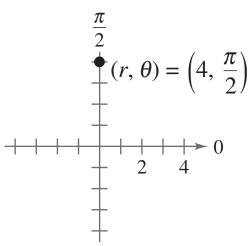
Vocabulary Check

Fill in the blanks.

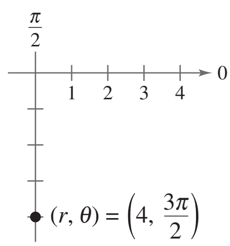
1. The origin of the polar coordinate system is called the ________ .
2. For the point (r, θ), r is the _______ from O to P and θ is the _______ counterclockwise from the polar axis to segment $\overline{OP}$.
3. To graph the point (r, θ), you use the _______ coordinate system.

In Exercises 1–4, a point in polar coordinates is given. Find the corresponding rectangular coordinates for the point.

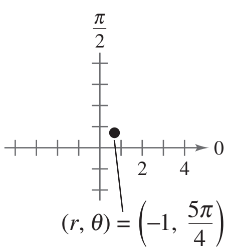
1. $\left(4, \dfrac{\pi}{2}\right)$



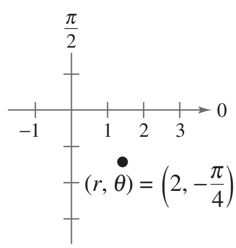
2. $\left(4, \dfrac{3\pi}{2}\right)$



3. $\left(-1, \dfrac{5\pi}{4}\right)$



4. $\left(2, -\dfrac{\pi}{4}\right)$



In Exercises 5–12, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.

5. $\left(3, \dfrac{5\pi}{6}\right)$

6. $\left(2, \dfrac{3\pi}{4}\right)$

7. $\left(-1, -\dfrac{\pi}{3}\right)$

8. $\left(-3, -\dfrac{7\pi}{6}\right)$

9. $\left(\sqrt{3}, \dfrac{5\pi}{6}\right)$

10. $\left(5\sqrt{2}, -\dfrac{11\pi}{6}\right)$

11. $\left(\dfrac{3}{2}, -\dfrac{3\pi}{2}\right)$

12. $\left(0, -\dfrac{\pi}{4}\right)$

In Exercises 13–20, plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

13. $\left(4, -\dfrac{\pi}{3}\right)$

14. $\left(2, \dfrac{7\pi}{6}\right)$

15. $\left(-1, -\dfrac{3\pi}{4}\right)$

16. $\left(-3, -\dfrac{2\pi}{3}\right)$

17. $\left(0, -\dfrac{7\pi}{6}\right)$

18. $\left(0, \dfrac{5\pi}{4}\right)$

19. $(\sqrt{2}, 2.36)$

20. $(-3, -1.57)$

In Exercises 21–28, use a graphing utility to find the rectangular coordinates of the point given in polar coordinates. Round your results to two decimal places.

21. $\left(2, \dfrac{2\pi}{9}\right)$

22. $\left(4, \dfrac{11\pi}{9}\right)$

23. $(-4.5, 1.3)$

24. $(8.25, 3.5)$

25. $(2.5, 1.58)$

26. $(5.4, 2.85)$

27. $(-4.1, -0.5)$

28. $(8.2, -3.2)$

In Exercises 29–36, plot the point given in rectangular coordinates and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

29. $(-7, 0)$

30. $(0, -5)$

31. $(1, 1)$

32. $(-3, -3)$

33. $(-\sqrt{3}, -\sqrt{3})$

34. $(\sqrt{3}, -1)$

35. $(6, 9)$

36. $(5, 12)$

In Exercises 37–42, use a graphing utility to find one set of polar coordinates for the point given in rectangular coordinates. (There are many correct answers.)

37. $(3, -2)$

38. $(-5, 2)$

39. $(\sqrt{3}, 2)$

40. $(3\sqrt{2}, 3\sqrt{2})$

41. $\left(\frac{5}{2}, \frac{4}{3}\right)$

42. $\left(\frac{7}{4}, \frac{3}{2}\right)$

In Exercises 43–60, convert the rectangular equation to polar form. Assume $a > 0$.

43. $x^2 + y^2 = 9$
44. $x^2 + y^2 = 16$
45. $y = 4$
46. $y = x$
47. $x = 8$
48. $x = a$
49. $3x - 6y + 2 = 0$
50. $4x + 7y - 2 = 0$
51. $xy = 4$
52. $2xy = 1$
53. $(x^2 + y^2)^2 = 9(x^2 - y^2)$
54. $y^2 - 8x - 16 = 0$
55. $x^2 + y^2 - 6x = 0$
56. $x^2 + y^2 - 8y = 0$
57. $x^2 + y^2 - 2ax = 0$
58. $x^2 + y^2 - 2ay = 0$
59. $y^2 = x^3$
60. $x^2 = y^3$

In Exercises 61–80, convert the polar equation to rectangular form.

61. $r = 6 \sin \theta$
62. $r = 2 \cos \theta$
63. $\theta = \dfrac{4\pi}{3}$
64. $\theta = \dfrac{5\pi}{3}$
65. $\theta = \dfrac{5\pi}{6}$
66. $\theta = \dfrac{11\pi}{6}$
67. $\theta = \dfrac{\pi}{2}$
68. $\theta = \pi$
69. $r = 4$
70. $r = 10$
71. $r = -3 \csc \theta$
72. $r = 2 \sec \theta$
73. $r^2 = \cos \theta$
74. $r^2 = \sin 2\theta$
75. $r = 2 \sin 3\theta$
76. $r = 3 \cos 2\theta$
77. $r = \dfrac{1}{1 - \cos \theta}$
78. $r = \dfrac{2}{1 + \sin \theta}$
79. $r = \dfrac{6}{2 - 3 \sin \theta}$
80. $r = \dfrac{6}{2 \cos \theta - 3 \sin \theta}$

In Exercises 81–86, describe the graph of the polar equation and find the corresponding rectangular equation. Sketch its graph.

81. $r = 7$
82. $r = 8$
83. $\theta = \dfrac{\pi}{4}$
84. $\theta = \dfrac{7\pi}{6}$
85. $r = 3 \sec \theta$
86. $r = 2 \csc \theta$

Synthesis

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. If (r_1, θ_1) and (r_2, θ_2) represent the same point in the polar coordinate system, then $|r_1| = |r_2|$.

88. If (r, θ_1) and (r, θ_2) represent the same point in the polar coordinate system, then $\theta_1 = \theta_2 + 2\pi n$ for some integer n.

89. ***Think About It***

(a) Show that the distance between the points (r_1, θ_1) and (r_2, θ_2) is

$$\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}.$$

(b) Describe the positions of the points relative to each other for $\theta_1 = \theta_2$. Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.

(c) Simplify the Distance Formula for $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.

(d) Choose two points in the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

90. ***Writing*** Write a short paragraph explaining the differences between the rectangular coordinate system and the polar coordinate system.

Skills Review

In Exercises 91–96, use the Law of Sines or the Law of Cosines to solve the triangle.

91. $a = 13, b = 19, c = 25$
92. $A = 24°, a = 10, b = 6$
93. $A = 56°, C = 38°, c = 12$
94. $B = 71°, a = 21, c = 29$
95. $C = 35°, a = 8, b = 4$
96. $B = 64°, b = 52, c = 44$

In Exercises 97–102, use any method to solve the system of equations.

97. $\begin{cases} 5x - 7y = -11 \\ -3x + y = -3 \end{cases}$

98. $\begin{cases} 3x + 5y = 10 \\ 4x - 2y = -5 \end{cases}$

99. $\begin{cases} 3a - 2b + c = 0 \\ 2a + b - 3c = 0 \\ a - 3b + 9c = 0 \end{cases}$

100. $\begin{cases} 5u + 7v + 9w = 15 \\ u - 2v - 3w = 7 \\ 8u - 2v + w = 0 \end{cases}$

101. $\begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$

102. $\begin{cases} 2x_1 + x_2 + 2x_3 = 4 \\ 2x_1 + 2x_2 = 5 \\ 2x_1 - x_2 + 6x_3 = 2 \end{cases}$

In Exercises 103–106, use a determinant to determine whether the points are collinear.

103. $(4, -3), (6, -7), (-2, -1)$
104. $(-2, 4), (0, 1), (4, -5)$
105. $(-6, -4), (-1, -3), (1.5, -2.5)$
106. $(-2.3, 5), (-0.5, 0), (1.5, -3)$

9.7 Graphs of Polar Equations

Introduction

In previous chapters you sketched graphs in rectangular coordinate systems. You began with the basic point-plotting method. Then you used sketching aids such as a graphing utility, symmetry, intercepts, asymptotes, periods, and shifts to further investigate the nature of the graph. This section approaches curve sketching in the polar coordinate system similarly.

What you should learn

- Graph polar equations by point plotting.
- Use symmetry as a sketching aid.
- Use zeros and maximum r-values as sketching aids.
- Recognize special polar graphs.

Why you should learn it

Several common figures, such as the circle in Exercise 4 on page 720, are easier to graph in the polar coordinate system than in the rectangular coordinate system.

Example 1 Graphing a Polar Equation by Point Plotting

Sketch the graph of the polar equation $r = 4 \sin \theta$ by hand.

Solution

The sine function is periodic, so you can get a full range of r-values by considering values of θ in the interval $0 \le \theta \le 2\pi$, as shown in the table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0

By plotting these points as shown in Figure 9.79, it appears that the graph is a circle of radius 2 whose center is the point $(x, y) = (0, 2)$.

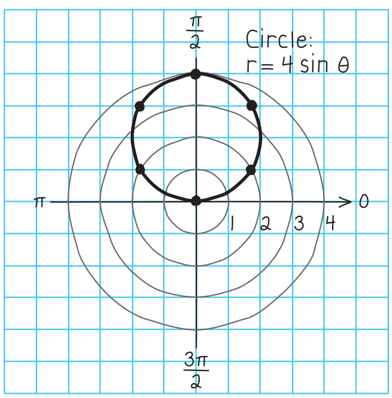


Figure 9.79

CHECKPOINT Now try Exercise 25.

Prerequisite Skills

If you have trouble finding the sines of the angles in Example 1, review Trigonometric Functions of Any Angle in Section 4.4.

You can confirm the graph found in Example 1 in three ways.

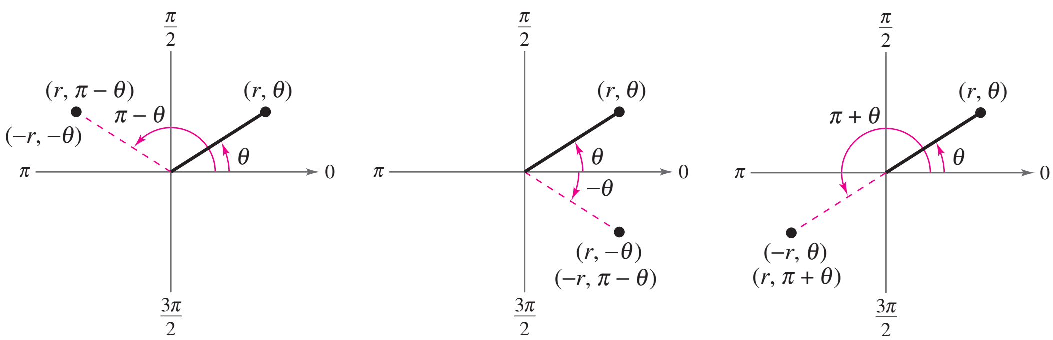
1. *Convert to Rectangular Form* Multiply each side of the polar equation by r and convert the result to rectangular form.
2. *Use a Polar Coordinate Mode* Set your graphing utility to *polar* mode and graph the polar equation. (Use $0 \le \theta \le \pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.)
3. *Use a Parametric Mode* Set your graphing utility to *parametric* mode and graph $x = (4 \sin t) \cos t$ and $y = (4 \sin t) \sin t$.

Most graphing utilities have a *polar* graphing mode. If yours doesn't, you can rewrite the polar equation $r = f(\theta)$ in parametric form, using t as a parameter, as follows.

$$x = f(t)\cos t \qquad \text{and} \qquad y = f(t)\sin t$$

Symmetry

In Figure 9.79, note that as θ increases from 0 to 2π the graph is traced out twice. Moreover, note that the graph is *symmetric with respect to the line* $\theta = \pi/2$. Had you known about this symmetry and retracing ahead of time, you could have used fewer points. The three important types of symmetry to consider in polar curve sketching are shown in Figure 9.80.



Symmetry with Respect to the Line $\boldsymbol{\theta = \frac{\pi}{2}}$

Symmetry with Respect to the Polar Axis

Symmetry with Respect to the Pole

Figure 9.80

Testing for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

1. The line $\theta = \dfrac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. The polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. The pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

You can determine the symmetry of the graph of $r = 4\sin\theta$ (see Example 1) as follows.

1. Replace (r, θ) by $(-r, -\theta)$:

$$-r = 4\sin(-\theta) \quad \Longrightarrow \quad r = -4\sin(-\theta) = 4\sin\theta$$

2. Replace (r, θ) by $(r, -\theta)$: $r = 4\sin(-\theta) = -4\sin\theta$
3. Replace (r, θ) by $(-r, \theta)$: $-r = 4\sin\theta \quad \Longrightarrow \quad r = -4\sin\theta$

So, the graph of $r = 4\sin\theta$ is symmetric with respect to the line $\theta = \pi/2$.

STUDY TIP

Recall from Section 4.2 that the sine function is odd. That is, $\sin(-\theta) = -\sin\theta$.

Example 2 Using Symmetry to Sketch a Polar Graph

Use symmetry to sketch the graph of $r = 3 + 2\cos\theta$ by hand.

Solution

Replacing (r, θ) by $(r, -\theta)$ produces

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta. \qquad \cos(-u) = \cos u$$

So, by using the even trigonometric identity, you can conclude that the curve is symmetric with respect to the polar axis. Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 9.81. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	5	$3 + \sqrt{3}$	4	3	2	$3 - \sqrt{3}$	1

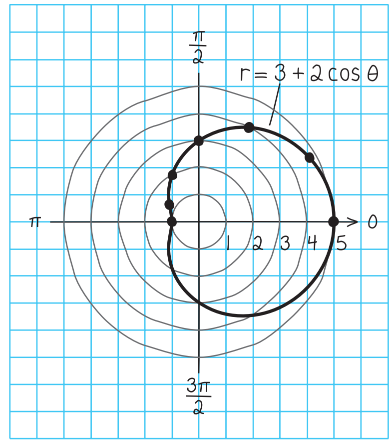


Figure 9.81

Use a graphing utility to confirm this graph.

CHECKPOINT Now try Exercise 29.

The three tests for symmetry in polar coordinates on page 714 are sufficient to guarantee symmetry, but they are not necessary. For instance, Figure 9.82 shows the graph of

$$r = \theta + 2\pi. \qquad \text{Spiral of Archimedes}$$

From the figure, you can see that the graph is symmetric with respect to the line $\theta = \pi/2$. Yet the tests on page 714 fail to indicate symmetry because neither of the following replacements yields an equivalent equation.

Original Equation	*Replacement*	*New Equation*
$r = \theta + 2\pi$	(r, θ) by $(-r, -\theta)$	$-r = -\theta + 2\pi$
$r = \theta + 2\pi$	(r, θ) by $(r, \pi - \theta)$	$r = -\theta + 3\pi$

The equations discussed in Examples 1 and 2 are of the form

$$r = 4\sin\theta = f(\sin\theta) \qquad \text{and} \qquad r = 3 + 2\cos\theta = g(\cos\theta).$$

The graph of the first equation is symmetric with respect to the line $\theta = \pi/2$, and the graph of the second equation is symmetric with respect to the polar axis. This observation can be generalized to yield the following *quick tests for symmetry*.

Quick Tests for Symmetry in Polar Coordinates

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.
2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the polar axis.

TECHNOLOGY TIP

The *table* feature of a graphing utility is very useful in constructing tables of values for polar equations. Set your graphing utility to *polar* mode and enter the polar equation in Example 2. You can verify the table of values in Example 2 by starting the table at $\theta = 0$ and incrementing the value of θ by $\pi/6$. For instructions on how to use the *table* feature and *polar* mode, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

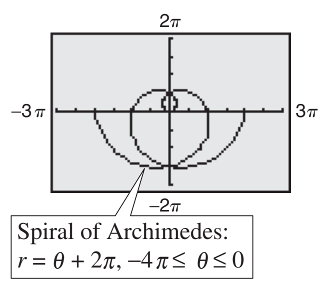


Figure 9.82

Zeros and Maximum r-Values

Two additional aids to sketching graphs of polar equations involve knowing the θ-values for which $|r|$ is maximum and knowing the θ-values for which $r = 0$. In Example 1, the maximum value of $|r|$ for $r = 4 \sin \theta$ is $|r| = 4$, and this occurs when $\theta = \pi/2$ (see Figure 9.79). Moreover, $r = 0$ when $\theta = 0$.

Example 3 Finding Maximum r-Values of a Polar Graph

Find the maximum value of r for the graph of $r = 1 - 2 \cos \theta$.

Graphical Solution

Because the polar equation is of the form

$$r = 1 - 2 \cos \theta = g(\cos \theta)$$

you know the graph is symmetric with respect to the polar axis. You can confirm this by graphing the polar equation. Set your graphing utility to *polar* mode and enter the equation, as shown in Figure 9.83. (In the graph, θ varies from 0 to 2π.) To find the maximum r-value for the graph, use your graphing utility's *trace* feature and you should find that the graph has a maximum r-value of 3, as shown in Figure 9.84. This value of r occurs when $\theta = \pi$. In the graph, note that the point $(3, \pi)$ is farthest from the pole.

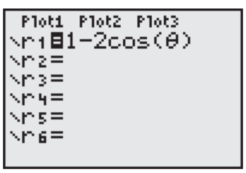


Figure 9.83

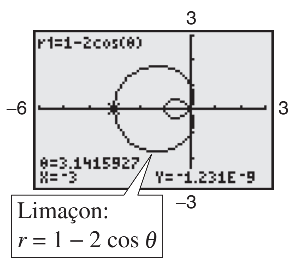


Figure 9.84

Note how the negative r-values determine the *inner loop* of the graph in Figure 9.84. This type of graph is a limaçon.

CHECKPOINT Now try Exercise 19.

Numerical Solution

To approximate the maximum value of r for the graph of $r = 1 - 2 \cos \theta$, use the *table* feature of a graphing utility to create a table that begins at $\theta = 0$ and increments by $\pi/12$, as shown in Figure 9.85. From the table, the maximum value of r appears to be 3 when $\theta = 3.1416 \approx \pi$. If a second table that begins at $\theta = \pi/2$ and increments by $\pi/24$ is created, as shown in Figure 9.86, the maximum value of r still appears to be 3 when $\theta = 3.1416 \approx \pi$.

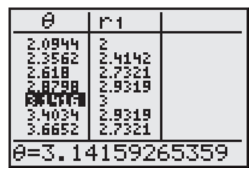


Figure 9.85

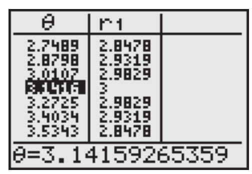


Figure 9.86

Exploration

The graph of the polar equation $r = e^{\cos \theta} - 2 \cos 4\theta + \sin^5(\theta/12)$ is called *the butterfly curve*, as shown in Figure 9.87.

a. The graph in Figure 9.87 was produced using $0 \le \theta \le 2\pi$. Does this show the entire graph? Explain your reasoning.

b. Use the *trace* feature of your graphing utility to approximate the maximum r-value of the graph. Does this value change if you use $0 \le \theta \le 4\pi$ instead of $0 \le \theta \le 2\pi$? Explain.

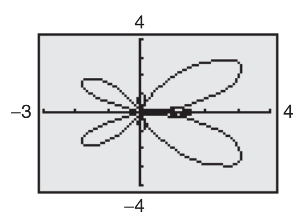


Figure 9.87

Some curves reach their zeros and maximum r-values at more than one point, as shown in Example 4.

Example 4 Analyzing a Polar Graph

Analyze the graph of $r = 2 \cos 3\theta$.

Solution

Symmetry: With respect to the polar axis

Maximum value of $|r|$: $|r| = 2$ when $3\theta = 0, \pi, 2\pi, 3\pi$
or $\theta = 0, \pi/3, 2\pi/3, \pi$

Zeros of r: $r = 0$ when $3\theta = \pi/2, 3\pi/2, 5\pi/2$
or $\theta = \pi/6, \pi/2, 5\pi/6$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0

By plotting these points and using the specified symmetry, zeros, and maximum values, you can obtain the graph shown in Figure 9.88. This graph is called a **rose curve,** and each loop on the graph is called a *petal*. Note how the entire curve is generated as θ increases from 0 to π.

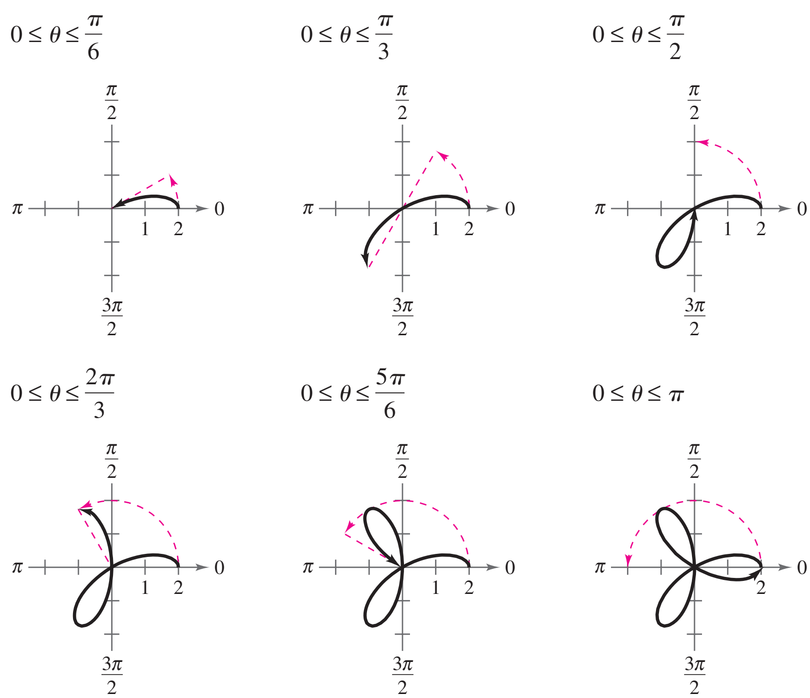


Figure 9.88

CHECKPOINT Now try Exercise 33.

Exploration

Notice that the rose curve in Example 4 has three petals. How many petals do the rose curves $r = 2 \cos 4\theta$ and $r = 2 \sin 3\theta$ have? Determine the numbers of petals for the curves $r = 2 \cos n\theta$ and $r = 2 \sin n\theta$, where n is a positive integer.

Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the circle

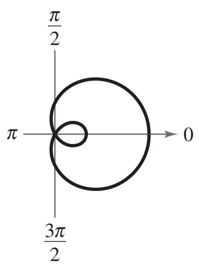
$$r = 4 \sin \theta$$

in Example 1 has the more complicated rectangular equation

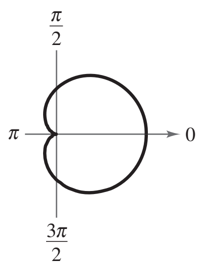
$$x^2 + (y - 2)^2 = 4.$$

Several other types of graphs that have simple polar equations are shown below.

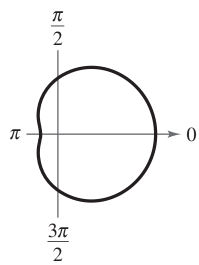
Limaçons
$r = a \pm b \cos \theta$
$r = a \pm b \sin \theta$
$(a > 0, b > 0)$



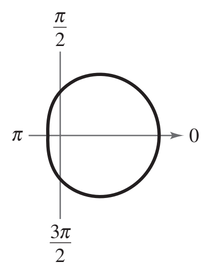
$\frac{a}{b} < 1$
Limaçon with inner loop



$\frac{a}{b} = 1$
Cardioid (heart-shaped)

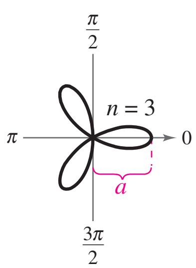


$1 < \frac{a}{b} < 2$
Dimpled limaçon

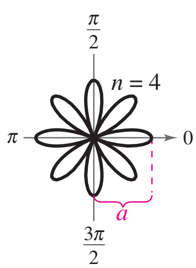


$\frac{a}{b} \geq 2$
Convex limaçon

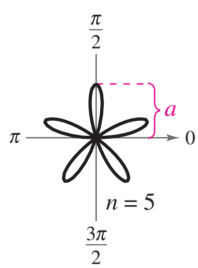
Rose Curves
n petals if n is odd
$2n$ petals if n is even
$(n \geq 2)$



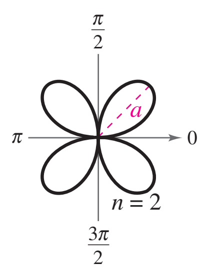
$r = a \cos n\theta$
Rose curve



$r = a \cos n\theta$
Rose curve

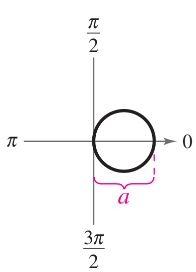


$r = a \sin n\theta$
Rose curve

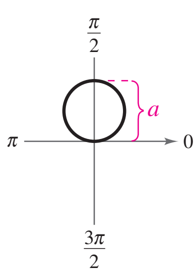


$r = a \sin n\theta$
Rose curve

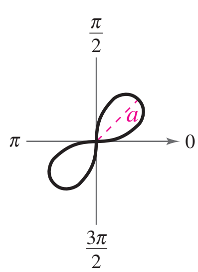
Circles and Lemniscates



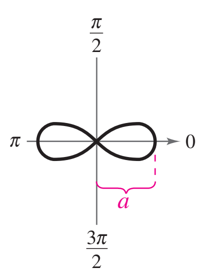
$r = a \cos \theta$
Circle



$r = a \sin \theta$
Circle



$r^2 = a^2 \sin 2\theta$
Lemniscate



$r^2 = a^2 \cos 2\theta$
Lemniscate

Example 5 Analyzing a Rose Curve

Analyze the graph of $r = 3 \cos 2\theta$.

Solution

Type of curve: Rose curve with $2n = 4$ petals

Symmetry: With respect to the polar axis, the line $\theta = \pi/2$, and the pole

Maximum value of $|r|$: $|r| = 3$ when $\theta = 0, \pi/2, \pi, 3\pi/2$

Zeros of r: $r = 0$ when $\theta = \pi/4, 3\pi/4$

Using a graphing utility, enter the equation, as shown in Figure 9.89 (with $0 \le \theta \le 2\pi$). You should obtain the graph shown in Figure 9.90.

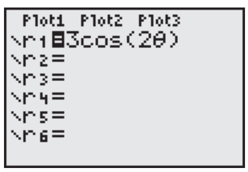


Figure 9.89

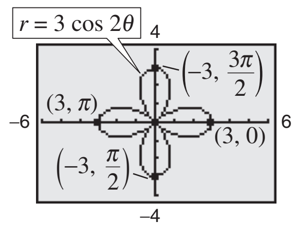


Figure 9.90

CHECKPOINT Now try Exercise 37.

> **STUDY TIP**
>
> The quick tests for symmetry presented on page 715 are especially useful when graphing rose curves. Because rose curves have the form $r = f(\sin \theta)$ or the form $r = g(\cos \theta)$, you know that a rose curve will be either symmetric with respect to the line $\theta = \pi/2$ or symmetric with respect to the polar axis.

Example 6 Analyzing a Lemniscate

Analyze the graph of $r^2 = 9 \sin 2\theta$.

Solution

Type of curve: Lemniscate

Symmetry: With respect to the pole

Maximum value of $|r|$: $|r| = 3$ when $\theta = \pi/4$

Zeros of r: $r = 0$ when $\theta = 0, \pi/2$

Using a graphing utility, enter the equation, as shown in Figure 9.91 (with $0 \le \theta \le 2\pi$). You should obtain the graph shown in Figure 9.92.

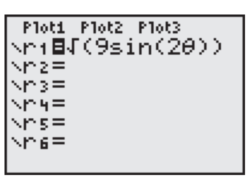


Figure 9.91

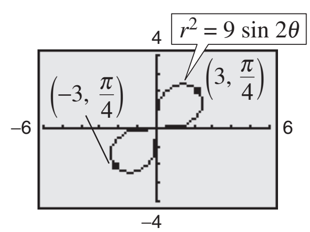


Figure 9.92

CHECKPOINT Now try Exercise 45.

9.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line ________ .

2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the ________ .

3. The equation $r = 2 + \cos\theta$ represents a ________ .

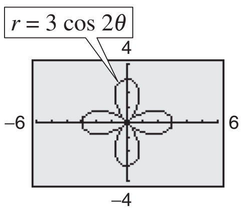
4. The equation $r = 2\cos\theta$ represents a ________ .

5. The equation $r^2 = 4\sin 2\theta$ represents a ________ .

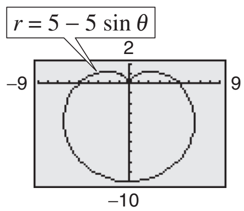
6. The equation $r = 1 + \sin\theta$ represents a ________ .

In Exercises 1–6, identify the type of polar graph.

1.



2.



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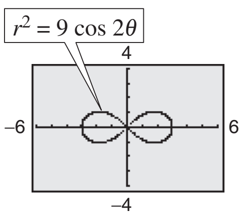

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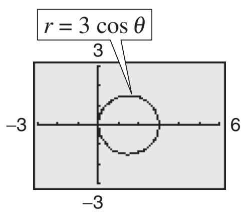

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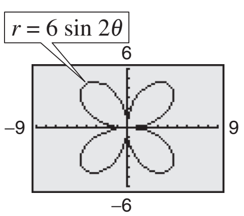

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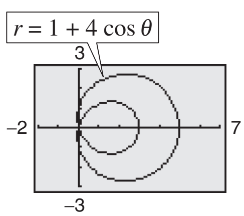
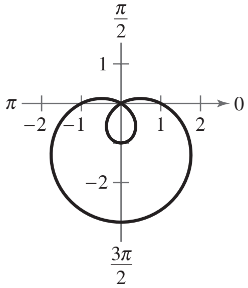


Library of Parent Functions **In Exercises 7–10, determine the equation of the polar curve whose graph is shown.**

7.



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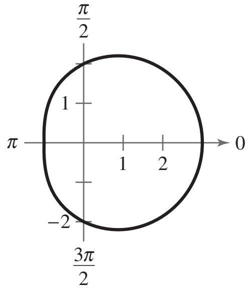


(a) $r = 1 - 2\sin\theta$

(b) $r = 1 + 2\sin\theta$

(c) $r = 1 + 2\cos\theta$

(d) $r = 1 - 2\cos\theta$

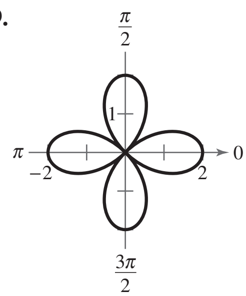
(a) $r = 2 - \cos\theta$

(b) $r = 2 - \sin\theta$

(c) $r = 2 + \cos\theta$

(d) $r = 2 + \sin\theta$

9.



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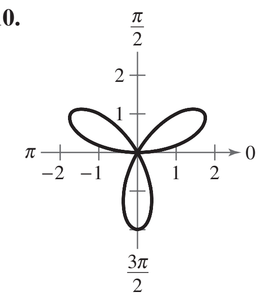


(a) $r = 2\cos 4\theta$

(b) $r = \cos 4\theta$

(c) $r = 2\cos 2\theta$

(d) $r = 2\cos\dfrac{\theta}{2}$

(a) $r = 2\sin 6\theta$

(b) $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

(c) $r = 2\sin\left(\dfrac{3\theta}{2}\right)$

(d) $r = 2\sin 3\theta$

In Exercises 11–18, test for symmetry with respect to $\theta = \pi/2$, the polar axis, and the pole.

11. $r = 14 + 4\cos\theta$

12. $r = 12\cos 3\theta$

13. $r = \dfrac{4}{1 + \sin\theta}$

14. $r = \dfrac{2}{1 - \cos\theta}$

15. $r = 6\sin\theta$

16. $r = 4\csc\theta\cos\theta$

17. $r^2 = 16\sin 2\theta$

18. $r^2 = 25\cos 4\theta$

In Exercises 19–22, find the maximum value of $|r|$ and any zeros of r. Verify your answers numerically.

19. $r = 10 - 10\sin\theta$

20. $r = 6 + 12\cos\theta$

21. $r = 4\cos 3\theta$

22. $r = \sin 2\theta$

In Exercises 23–36, sketch the graph of the polar equation. Use a graphing utility to verify your graph.

23. $r = 5$

24. $\theta = -\dfrac{5\pi}{3}$

25. $r = 3\sin\theta$

26. $r = 2\cos\theta$

27. $r = 3(1 - \cos\theta)$

28. $r = 4(1 + \sin\theta)$

29. $r = 3 - 4\cos\theta$

30. $r = 1 - 2\sin\theta$

31. $r = 4 + 5\sin\theta$

32. $r = 3 + 6\cos\theta$

33. $r = 5\cos 3\theta$

34. $r = -\sin 5\theta$

35. $r = 7\sin 2\theta$

36. $r = 3\cos 5\theta$

In Exercises 37–52, use a graphing utility to graph the polar equation. Describe your viewing window.

37. $r = 8\cos 2\theta$

38. $r = \cos 2\theta$

39. $r = 2(5 - \sin\theta)$

40. $r = 6 - 4\sin\theta$

41. $r = 3 - 6\cos\theta$

42. $r = 3 - 2\sin\theta$

43. $r = \dfrac{3}{\sin\theta - 2\cos\theta}$

44. $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

45. $r^2 = 4\cos 2\theta$

46. $r^2 = 9\sin\theta$

47. $r = 4\sin\theta\cos^2\theta$

48. $r = 2\cos(3\theta - 2)$

49. $r = 2\csc\theta + 6$

50. $r = 4 - \sec\theta$

51. $r = e^{2\theta}$

52. $r = e^{\theta/2}$

In Exercises 53–58, use a graphing utility to graph the polar equation. Find an interval for θ for which the graph is traced *only once*.

53. $r = 3 - 2\cos\theta$

54. $r = 2(1 - 2\sin\theta)$

55. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

56. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

57. $r^2 = \sin 2\theta$

58. $r^2 = \dfrac{1}{\theta}$

In Exercises 59–62, use a graphing utility to graph the polar equation and show that the indicated line is an asymptote of the graph.

	Name of Graph	*Polar Equation*	*Asymptote*
59.	Conchoid	$r = 2 - \sec\theta$	$x = -1$
60.	Conchoid	$r = 2 + \csc\theta$	$y = 1$
61.	Hyperbolic spiral	$r = 2/\theta$	$y = 2$
62.	Strophoid	$r = 2\cos 2\theta\sec\theta$	$x = -2$

Synthesis

True or False? **In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.**

63. The graph of $r = 10\sin 5\theta$ is a rose curve with five petals.

64. A rose curve will always have symmetry with respect to the line $\theta = \pi/2$.

65. ***Writing*** Use a graphing utility to graph the polar equation

$$r = \cos 5\theta + n\cos\theta, \quad 0 \le \theta < \pi$$

for the integers $n = -5$ to $n = 5$. As you graph these equations, you should see the graph change shape from a heart to a bell. Write a short paragraph explaining which values of n produce the heart-shaped curves and which values of n produce the bell-shaped curves.

66. The graph of $r = f(\theta)$ is rotated about the pole through an angle ϕ. Show that the equation of the rotated graph is $r = f(\theta - \phi)$.

67. Consider the graph of $r = f(\sin\theta)$.

(a) Show that if the graph is rotated counterclockwise $\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(-\cos\theta)$.

(b) Show that if the graph is rotated counterclockwise π radians about the pole, the equation of the rotated graph is $r = f(-\sin\theta)$.

(c) Show that if the graph is rotated counterclockwise $3\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(\cos\theta)$.

In Exercises 68–70, use the results of Exercises 66 and 67.

68. Write an equation for the limaçon $r = 2 - \sin\theta$ after it has been rotated through each given angle.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{3\pi}{2}$

69. Write an equation for the rose curve $r = 2\sin 2\theta$ after it has been rotated through each given angle.

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{2\pi}{3}$ (d) π

70. Sketch the graph of each equation.

(a) $r = 1 - \sin\theta$ (b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

71. ***Exploration*** Use a graphing utility to graph the polar equation $r = 2 + k\cos\theta$ for $k = 0$, $k = 1$, $k = 2$, and $k = 3$. Identify each graph.

72. ***Exploration*** Consider the polar equation $r = 3\sin k\theta$.

(a) Use a graphing utility to graph the equation for $k = 1.5$. Find the interval for θ for which the graph is traced only once.

(b) Use a graphing utility to graph the equation for $k = 2.5$. Find the interval for θ for which the graph is traced only once.

(c) Is it possible to find an interval for θ for which the graph is traced only once for any rational number k? Explain.

9.8 Polar Equations of Conics

Alternative Definition of Conics

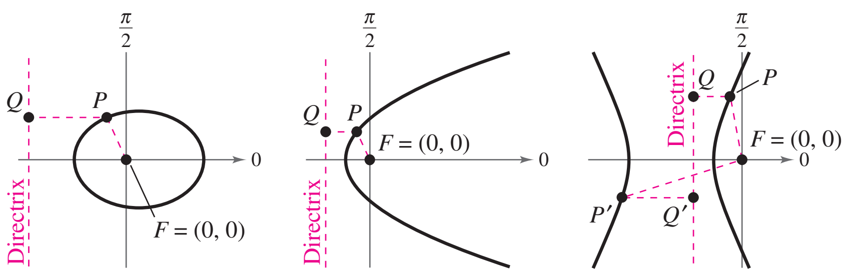
In Sections 9.2 and 9.3, you learned that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at the *center*. As it happens, there are many important applications of conics in which it is more convenient to use one of the *foci* as the origin. In this section, you will learn that polar equations of conics take simple forms if one of the foci lies at the pole.

To begin, consider the following alternative definition of a conic that uses the concept of eccentricity (a measure of the flatness of the conic).

Alternative Definition of a Conic

The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the **eccentricity** of the conic and is denoted by e. Moreover, the conic is an **ellipse** if $e < 1$, a **parabola** if $e = 1$, and a **hyperbola** if $e > 1$. (See Figure 9.93.)

In Figure 9.93, note that for each type of conic, the focus is at the pole.



Ellipse: $0 < e < 1$ $\quad \frac{PF}{PQ} < 1$

Parabola: $e = 1$ $\quad \frac{PF}{PQ} = 1$

Hyperbola: $e > 1$ $\quad \frac{PF}{PQ} = \frac{P'F'}{P'Q'} > 1$

Figure 9.93

Polar Equations of Conics

The benefit of locating a focus of a conic at the pole is that the equation of the conic becomes simpler.

Polar Equations of Conics (See the proof on page 739.)

The graph of a polar equation of the form

$$\textbf{1. } r = \frac{ep}{1 \pm e\cos\theta} \quad \text{or} \quad \textbf{2. } r = \frac{ep}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

What you should learn

- Define conics in terms of eccentricities.
- Write and graph equations of conics in polar form.
- Use equations of conics in polar form to model real-life problems.

Why you should learn it

The orbits of planets and satellites can be modeled by polar equations. For instance, in Exercise 55 on page 727, you will use polar equations to model the orbits of Neptune and Pluto.

Kevin Kelley/Getty Images

Prerequisite Skills

To review the characteristics of conics, see Sections 9.1–9.3.

An equation of the form

$$r = \frac{ep}{1 \pm e \cos \theta}$$ Vertical directrix

corresponds to a conic with a vertical directrix and symmetry with respect to the polar axis. An equation of the form

$$r = \frac{ep}{1 \pm e \sin \theta}$$ Horizontal directrix

corresponds to a conic with a horizontal directrix and symmetry with respect to $\theta = \pi/2$. Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of the given equations.

Example 1 Identifying a Conic from Its Equation

Identify the type of conic represented by the equation $r = \dfrac{15}{3 - 2 \cos \theta}$.

Algebraic Solution

To identify the type of conic, rewrite the equation in the form $r = ep/(1 \pm e \cos \theta)$.

$$r = \frac{15}{3 - 2 \cos \theta}$$

$$= \frac{5}{1 - (2/3) \cos \theta}$$ Divide numerator and denominator by 3.

Because $e = \frac{2}{3} < 1$, you can conclude that the graph is an ellipse.

CHECKPOINT Now try Exercise 11.

Graphical Solution

Use a graphing utility in *polar* mode to graph $r = \dfrac{15}{3 - 2 \cos \theta}$.

Be sure to use a square setting. From the graph in Figure 9.94, you can see that the conic appears to be an ellipse.

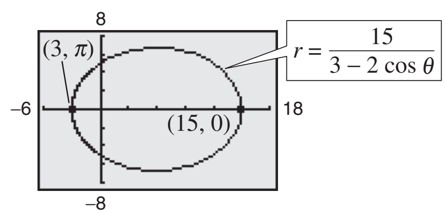


Figure 9.94

For the ellipse in Figure 9.94, the major axis is horizontal and the vertices lie at $(r, \theta) = (15, 0)$ and $(r, \theta) = (3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the *minor* axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2$$

$$= a^2 - (ea)^2$$

$$= a^2(1 - e^2).$$ Ellipse

Because $e = \frac{2}{3}$, you have $b^2 = 9^2\left[1 - \left(\frac{2}{3}\right)^2\right] = 45$, which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2$$

$$= (ea)^2 - a^2$$

$$= a^2(e^2 - 1).$$ Hyperbola

Example 2 Analyzing the Graph of a Polar Equation

Analyze the graph of the polar equation

$$r = \frac{32}{3 + 5\sin\theta}.$$

Solution

Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$ and the vertices occur at $(r, \theta) = (4, \pi/2)$ and $(r, \theta) = (-16, 3\pi/2)$. Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

So, $b = 8$. You can use a and b to determine that the asymptotes are $y = 10 \pm \frac{3}{4}x$, as shown in Figure 9.95.

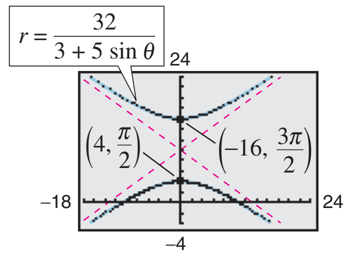


Figure 9.95

CHECKPOINT Now try Exercise 23.

Exploration

Try using a graphing utility in *polar* mode to verify the four orientations shown at the left. Remember that e must be positive, but p can be positive or negative.

In the next example, you are asked to find a polar equation for a specified conic. To do this, let p be the distance between the pole and the directrix.

1. Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e\sin\theta}$
2. Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e\sin\theta}$
3. Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e\cos\theta}$
4. Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e\cos\theta}$

Example 3 Finding the Polar Equation of a Conic

Find the polar equation of the parabola whose focus is the pole and whose directrix is the line $y = 3$.

Solution

From Figure 9.96, you can see that the directrix is horizontal and above the pole. Moreover, because the eccentricity of a parabola is $e = 1$ and the distance between the pole and the directrix is $p = 3$, you have the equation

$$r = \frac{ep}{1 + e\sin\theta} = \frac{3}{1 + \sin\theta}.$$

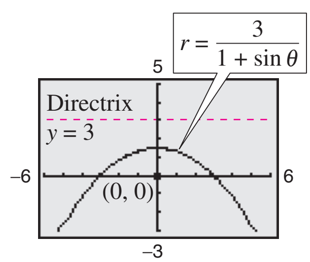


Figure 9.96

CHECKPOINT Now try Exercise 33.

Application

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period (the time it takes for a planet to orbit the sun) is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler simply stated these laws on the basis of observation, they were later validated by Isaac Newton (1642–1727). In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is illustrated in the next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

If you use Earth as a reference with a period of 1 year and a distance of 1 astronomical unit (an *astronomical unit* is defined as the mean distance between Earth and the sun, or about 93 million miles), the proportionality constant in Kepler's third law is 1. For example, because Mars has a mean distance to the sun of $d \approx 1.524$ astronomical units, its period P is given by $d^3 = P^2$. So, the period of Mars is $P \approx 1.88$ years.

Example 4 Halley's Comet

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

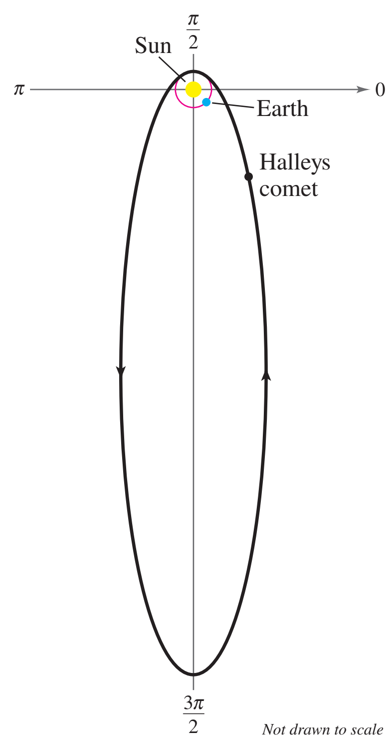


Figure 9.97

Solution

Using a vertical axis, as shown in Figure 9.97, choose an equation of the form $r = ep/(1 + e \sin \theta)$. Because the vertices of the ellipse occur at $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices. That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88.$$

So, $p \approx 1.204$ and $ep \approx (0.967)(1.204) \approx 1.164$. Using this value of ep in the equation, you have

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.164}{1 + 0.967 \sin(\pi/2)} \approx 0.59 \text{ astronomical units} \approx 55{,}000{,}000 \text{ miles}.$$

CHECKPOINT Now try Exercise 51.

9.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

In Exercises 1 and 2, fill in the blanks.

1. The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a ________ .

2. The constant ratio is the ________ of the conic and is denoted by ________ .

3. Match the conic with its eccentricity.

(a) $e < 1$ (i) ellipse

(b) $e = 1$ (ii) hyperbola

(c) $e > 1$ (iii) parabola

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the polar equation for (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$. Identify the conic for each equation.**

1. $r = \dfrac{2e}{1 + e\cos\theta}$

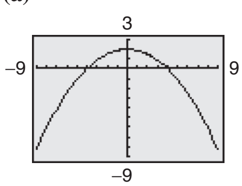
2. $r = \dfrac{2e}{1 - e\cos\theta}$

3. $r = \dfrac{2e}{1 - e\sin\theta}$

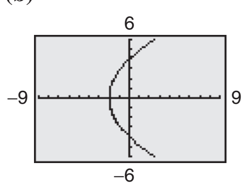
4. $r = \dfrac{2e}{1 + e\sin\theta}$

In Exercises 5–10, match the polar equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

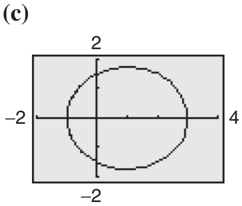
(a)



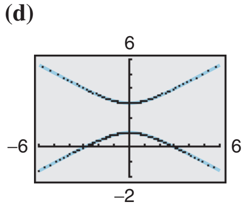
(b)



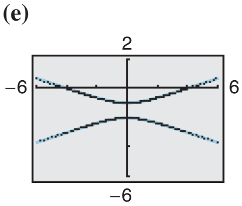
(c)



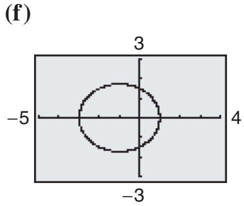
(d)



(e)



(f)



5. $r = \dfrac{4}{1 - \cos\theta}$

6. $r = \dfrac{3}{2 - \cos\theta}$

7. $r = \dfrac{3}{2 + \cos\theta}$

8. $r = \dfrac{4}{1 - 3\sin\theta}$

9. $r = \dfrac{3}{1 + 2\sin\theta}$

10. $r = \dfrac{4}{1 + \sin\theta}$

In Exercises 11–20, identify the conic represented by the equation algebraically. Use a graphing utility to confirm your result.

11. $r = \dfrac{2}{1 - \cos\theta}$

12. $r = \dfrac{2}{1 + \sin\theta}$

13. $r = \dfrac{4}{4 - \cos\theta}$

14. $r = \dfrac{7}{7 + \sin\theta}$

15. $r = \dfrac{8}{4 + 3\sin\theta}$

16. $r = \dfrac{6}{3 - 2\cos\theta}$

17. $r = \dfrac{6}{2 + \sin\theta}$

18. $r = \dfrac{5}{-1 + 2\cos\theta}$

19. $r = \dfrac{3}{4 - 8\cos\theta}$

20 $r = \dfrac{10}{3 + 9\sin\theta}$

In Exercises 21–26, use a graphing utility to graph the polar equation. Identify the graph.

21. $r = \dfrac{-5}{1 - \sin\theta}$

22. $r = \dfrac{-1}{2 + 4\sin\theta}$

23. $r = \dfrac{14}{14 + 17\sin\theta}$

24. $r = \dfrac{12}{2 - \cos\theta}$

25. $r = \dfrac{3}{-4 + 2\cos\theta}$

26. $r = \dfrac{4}{1 - 2\cos\theta}$

In Exercises 27–32, use a graphing utility to graph the rotated conic.

27. $r = \dfrac{2}{1 - \cos(\theta - \pi/4)}$ (See Exercise 11.)

28. $r = \dfrac{7}{7 + \sin(\theta - \pi/3)}$ (See Exercise 14.)

29. $r = \dfrac{4}{4 - \cos(\theta + 3\pi/4)}$ (See Exercise 13.)

30. $r = \dfrac{6}{3 - 2\cos(\theta + \pi/2)}$ (See Exercise 16.)

31. $r = \dfrac{8}{4 + 3\sin(\theta + \pi/6)}$ (See Exercise 15.)

32. $r = \dfrac{5}{-1 + 2\cos(\theta + 2\pi/3)}$ (See Exercise 18.)

In Exercises 33–48, find a polar equation of the conic with its focus at the pole.

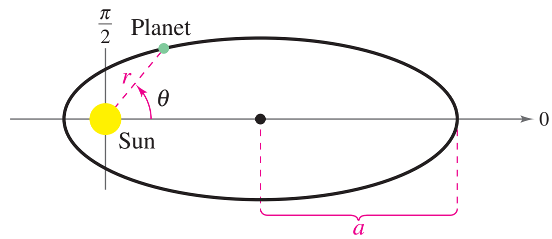
Conic	*Eccentricity*	*Directrix*
33. Parabola	$e = 1$	$x = -1$
34. Parabola	$e = 1$	$y = -4$
35. Ellipse	$e = \frac{1}{2}$	$y = 1$
36. Ellipse	$e = \frac{3}{4}$	$y = -4$
37. Hyperbola	$e = 2$	$x = 1$
38. Hyperbola	$e = \frac{3}{2}$	$x = -1$

Conic	*Vertex or Vertices*
39. Parabola	$\left(1, -\dfrac{\pi}{2}\right)$
40. Parabola	$(8, 0)$
41. Parabola	$(5, \pi)$
42. Parabola	$\left(10, \dfrac{\pi}{2}\right)$
43. Ellipse	$(2, 0), (10, \pi)$
44. Ellipse	$\left(2, \dfrac{\pi}{2}\right), \left(4, \dfrac{3\pi}{2}\right)$
45. Ellipse	$(20, 0), (4, \pi)$
46. Hyperbola	$\left(1, \dfrac{3\pi}{2}\right), \left(9, \dfrac{3\pi}{2}\right)$
47. Hyperbola	$\left(4, \dfrac{\pi}{2}\right), \left(-1, \dfrac{3\pi}{2}\right)$
48. Hyperbola	$\left(4, \dfrac{\pi}{2}\right), \left(1, \dfrac{\pi}{2}\right)$

49. ***Planetary Motion*** The planets travel in elliptical orbits with the sun at one focus. Assume that the focus is at the pole, the major axis lies on the polar axis, and the length of the major axis is $2a$ (see figure). Show that the polar equation of the orbit of a planet is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity.



50. ***Planetary Motion*** Use the result of Exercise 49 to show that the minimum distance (*perihelion*) from the sun to a planet is $r = a(1 - e)$ and that the maximum distance (*aphelion*) is $r = a(1 + e)$.

Planetary Motion **In Exercises 51–54, use the results of Exercises 49 and 50 to find the polar equation of the orbit of the planet and the perihelion and aphelion distances.**

51. Earth $a = 92.956 \times 10^6$ miles
$e = 0.0167$

52. Mercury $a = 35.983 \times 10^6$ miles
$e = 0.2056$

53. Jupiter $a = 77.841 \times 10^7$ kilometers
$e = 0.0484$

54. Saturn $a = 142.673 \times 10^7$ kilometers
$e = 0.0542$

55. ***Planetary Motion*** Use the results of Exercises 49 and 50, where for the planet Neptune, $a = 4.498 \times 10^9$ kilometers and $e = 0.0086$ and for the dwarf planet Pluto, $a = 5.906 \times 10^9$ kilometers and $e = 0.2488$.

(a) Find the polar equation of the orbit of each planet.

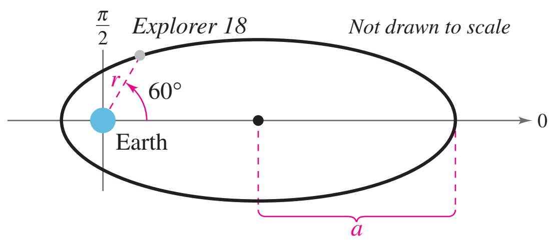
(b) Find the perihelion and aphelion distances for each planet.

(c) Use a graphing utility to graph both Neptune's and Pluto's equations of orbit in the same viewing window.

(d) Is Pluto ever closer to the sun than Neptune? Until recently, Pluto was considered the ninth planet. Why was Pluto called the ninth planet and Neptune the eighth planet?

(e) Do the orbits of Neptune and Pluto intersect? Will Neptune and Pluto ever collide? Why or why not?

56. ***Explorer 18*** On November 27, 1963, the United States launched a satellite named *Explorer 18*. Its low and high points above the surface of Earth were 119 miles and 122,800 miles, respectively (see figure). The center of Earth is at one focus of the orbit.



(a) Find the polar equation for the orbit (assume the radius of Earth is 4000 miles).

(b) Find the distance between the surface of Earth and the satellite when $\theta = 60°$.

(c) Find the distance between the surface of Earth and the satellite when $\theta = 30°$.

Synthesis

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. Justify your answer.**

57. The graph of $r = 4/(-3 - 3 \sin \theta)$ has a horizontal directrix above the pole.

58. The conic represented by the following equation is an ellipse.

$$r^2 = \frac{16}{9 - 4\cos\left(\theta + \frac{\pi}{4}\right)}$$

59. Show that the polar equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}.$$

60. Show that the polar equation for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{-b^2}{1 - e^2 \cos^2 \theta}.$$

In Exercises 61–66, use the results of Exercises 59 and 60 to write the polar form of the equation of the conic.

61. $\frac{x^2}{169} + \frac{y^2}{144} = 1$

62. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

63. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

64. $\frac{x^2}{36} - \frac{y^2}{4} = 1$

65. Hyperbola One focus: $(5, \pi/2)$
Vertices: $(4, \pi/2), (4, -\pi/2)$

66. Ellipse One focus: $(4, 0)$
Vertices: $(5, 0), (5, \pi)$

67. ***Exploration*** Consider the polar equation

$$r = \frac{4}{1 - 0.4 \cos \theta}.$$

(a) Identify the conic without graphing the equation.

(b) Without graphing the following polar equations, describe how each differs from the given polar equation. Use a graphing utility to verify your results.

$$r = \frac{4}{1 + 0.4 \cos \theta}, \quad r = \frac{4}{1 - 0.4 \sin \theta}$$

68. ***Exploration*** The equation

$$r = \frac{ep}{1 \pm e \sin \theta}$$

is the equation of an ellipse with $e < 1$. What happens to the lengths of both the major axis and the minor axis when the value of e remains fixed and the value of p changes? Use an example to explain your reasoning.

69. ***Writing*** In your own words, define the term *eccentricity* and explain how it can be used to classify conics.

70. What conic does the polar equation given by $r = a \sin \theta + b \cos \theta$ represent?

Skills Review

In Exercises 71–76, solve the equation.

71. $4\sqrt{3} \tan \theta - 3 = 1$

72. $6 \cos x - 2 = 1$

73. $12 \sin^2 \theta = 9$

74. $9 \csc^2 x - 10 = 2$

75. $2 \cot x = 5 \cos \frac{\pi}{2}$

76. $\sqrt{2} \sec \theta = 2 \csc \frac{\pi}{4}$

In Exercises 77–80 find the value of the trigonometric function given that u and v are in Quadrant IV and $\sin u = -\frac{3}{5}$ and $\cos v = 1/\sqrt{2}$.

77. $\cos(u + v)$

78. $\sin(u + v)$

79. $\sin(u - v)$

80. $\cos(u - v)$

In Exercises 81–84, evaluate the expression. Do not use a calculator.

81. ${}_{12}C_9$

82. ${}_{18}C_{16}$

83. ${}_{10}P_3$

84. ${}_{29}P_2$

What Did You Learn?

Key Terms

conic (conic section), *p. 660*
degenerate conic, *p. 660*
circle, *p. 661*
center (of a circle), *p. 661*
radius, *p. 661*
parabola, *p. 663*
directrix, *p. 663*
focus (of a parabola), *p. 663*
vertex (of a parabola), *p. 663*
axis (of a parabola), *p. 663*
ellipse, *p. 671*
foci (of an ellipse), *p. 671*
vertices (of an ellipse), *p. 671*
major axis (of an ellipse), *p. 671*
center (of an ellipse), *p. 671*
minor axis (of an ellipse), *p. 671*
hyperbola, *p. 680*
foci (of a hyperbola), *p. 680*
branches, *p. 680*
vertices (of a hyperbola), *p. 680*
transverse axis, *p. 680*
center (of a hyperbola), *p. 680*
conjugate axis, *p. 682*
asymptotes (of a hyperbola), *p. 682*
rotation of axes, *p. 690*
invariant under rotation, *p. 694*
discriminant, *p. 694*
parameter, *p. 699*
parametric equations, *p. 699*
plane curve, *p. 699*
orientation, *p. 700*
eliminating the parameter, *p. 702*
polar coordinate system, *p. 707*
pole, *p. 707*
polar axis, *p. 707*
polar coordinates, *p. 707*
limaçon, *p. 715*
rose curve, *p. 717*
eccentricity, *p. 722*

Key Concepts

9.1–9.3 ■ Write and graph translations of conics

1. Circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

2. Parabola with vertex (h, k):

$$(x - h)^2 = 4p(y - k) \qquad \text{Vertical axis}$$

$$(y - k)^2 = 4p(x - h) \qquad \text{Horizontal axis}$$

3. Ellipse with center (h, k) and $0 < b < a$:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad \text{Horizontal major axis}$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \qquad \text{Vertical major axis}$$

4. Hyperbola with center (h, k):

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \qquad \text{Horizontal transverse axis}$$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \qquad \text{Vertical transverse axis}$$

9.4 ■ Rotation and systems of quadratic equations

1. Rotate the coordinate axis to eliminate the xy-term.

2. Use the discriminant to classify conics.

9.5 ■ Graph curves that are represented by sets of parametric equations

1. Sketch a curve represented by a pair of parametric equations by plotting points in the order of increasing values of t in the xy-plane.

2. Use the *parametric* mode of a graphing utility to graph a set of parametric equations.

9.6 ■ Convert points from rectangular to polar form and vice versa

1. Polar-to-Rectangular: $x = r\cos\theta,\ y = r\sin\theta$

2. Rectangular-to-Polar: $\tan\theta = \dfrac{x}{y},\ r^2 = x^2 + y^2$

9.7 ■ Use symmetry to aid in sketching graphs of polar equations

1. Symmetry with respect to the line $\theta = \dfrac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$

2. Symmetry with respect to the polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$

3. Symmetry with respect to the pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$

4. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

5. The graph of $r = g(\cos\theta)$ is symmetric with respect to the polar axis.

9.8 ■ Write and graph equations of conics in polar form

The graph of a polar equation of the form

$$r = \frac{ep}{1 \pm e\cos\theta} \text{ or } r = \frac{ep}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

9.1 In Exercises 1–4, find the standard form of the equation of the circle with the given characteristics.

1. Center at origin; point on the circle: $(-3, -4)$

2. Center at origin; point on the circle: $(8, -15)$

3. Endpoints of a diameter: $(-1, 2)$ and $(5, 6)$

4. Endpoints of a diameter: $(-2, 3)$ and $(6, -5)$

In Exercises 5–8, write the equation of the circle in standard form. Then identify its center and radius.

5. $\frac{1}{2}x^2 + \frac{1}{2}y^2 = 18$ **6.** $\frac{3}{4}x^2 + \frac{3}{4}y^2 = 1$

7. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

8. $4x^2 + 4y^2 + 32x - 24y + 51 = 0$

In Exercises 9 and 10, sketch the circle. Identify its center and radius.

9. $x^2 + y^2 + 4x + 6y - 3 = 0$

10. $x^2 + y^2 + 8x - 10y - 8 = 0$

In Exercises 11 and 12, find the *x*- and *y*-intercepts of the graph of the circle.

11. $(x - 3)^2 + (y + 1)^2 = 7$

12. $(x + 5)^2 + (y - 6)^2 = 27$

In Exercises 13–16, find the vertex, focus, and directrix of the parabola, and sketch its graph. Use a graphing utility to verify your graph.

13. $4x - y^2 = 0$ **14.** $y = -\frac{1}{8}x^2$

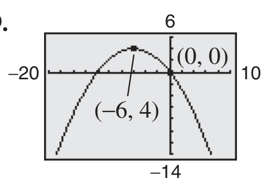
15. $\frac{1}{2}y^2 + 18x = 0$ **16.** $\frac{1}{4}y - 8x^2 = 0$

In Exercises 17–20, find the standard form of the equation of the parabola with the given characteristics.

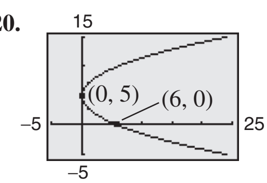
17. Vertex: $(0, 0)$
Focus: $(-6, 0)$

18. Vertex: $(4, 2)$
Focus: $(4, 0)$

19.



20.



In Exercises 21 and 22, find an equation of the tangent line to the parabola at the given point and find the *x*-intercept of the line.

21. $x^2 = -2y,\ (2, -2)$ **22.** $y^2 = -2x,\ (-8, -4)$

23. ***Architecture*** A parabolic archway (see figure) is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters. How wide is the archway at ground level?

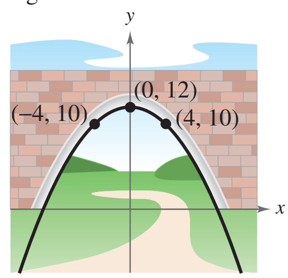


Figure for 23

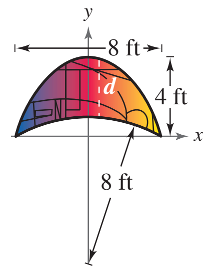


Figure for 24

24. ***Architecture*** A church window (see figure) is bounded on top by a parabola and below by the arc of a circle.

(a) Find equations of the parabola and the circle.

(b) Use a graphing utility to create a table showing the vertical distances d between the circle and the parabola for various values of x.

x	0	1	2	3	4
d					

9.2 In Exercises 25–28, find the center, vertices, foci, and eccentricity of the ellipse and sketch its graph. Use a graphing utility to verify your graph.

25. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$ **26.** $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

27. $\dfrac{(x - 4)^2}{6} + \dfrac{(y + 4)^2}{9} = 1$ **28.** $\dfrac{(x + 1)^2}{16} + \dfrac{(y - 3)^2}{6} = 1$

In Exercises 29–32, (a) find the standard form of the equation of the ellipse, (b) find the center, vertices, foci, and eccentricity of the ellipse, and (c) sketch the ellipse. Use a graphing utility to verify your graph.

29. $16x^2 + 9y^2 - 32x + 72y + 16 = 0$

30. $4x^2 + 25y^2 + 16x - 150y + 141 = 0$

31. $3x^2 + 8y^2 + 12x - 112y + 403 = 0$

32. $x^2 + 20y^2 - 5x + 120y + 185 = 0$

In Exercises 33–36, find the standard form of the equation of the ellipse with the given characteristics.

33. Vertices: $(\pm 5, 0)$; foci: $(\pm 4, 0)$

34. Vertices: $(0, \pm 6)$; passes through the point $(2, 2)$

35. Vertices: $(-3, 0), (7, 0)$; foci: $(0, 0), (4, 0)$

36. Vertices: $(2, 0), (2, 4)$; foci: $(2, 1), (2, 3)$

37. ***Architecture*** A semielliptical archway is to be formed over the entrance to an estate. The arch is to be set on pillars that are 10 feet apart and is to have a height (atop the pillars) of 4 feet. Where should the foci be placed in order to sketch the arch?

38. ***Wading Pool*** You are building a wading pool that is in the shape of an ellipse. Your plans give an equation for the elliptical shape of the pool measured in feet as

$$\frac{x^2}{324} + \frac{y^2}{196} = 1.$$

Find the longest distance across the pool, the shortest distance, and the distance between the foci.

39. ***Planetary Motion*** Saturn moves in an elliptical orbit with the sun at one focus. The smallest distance and the greatest distance of the planet from the sun are 1.3495×10^9 and 1.5045×10^9 kilometers, respectively. Find the eccentricity of the orbit, defined by $e = c/a$.

40. ***Planetary Motion*** Mercury moves in an elliptical orbit with the sun at one focus. The eccentricity of Mercury's orbit is $e = 0.2056$. The length of the major axis is 72 million miles. Find the standard equation of Mercury's orbit. Place the center of the orbit at the origin and the major axis on the x-axis.

9.3 In Exercises 41–46, (a) find the standard form of the equation of the hyperbola, (b) find the center, vertices, foci, and eccentricity of the hyperbola, and (c) sketch the hyperbola. Use a graphing utility to verify your graph in part (c).

41. $5y^2 - 4x^2 = 20$

42. $x^2 - y^2 = \frac{9}{4}$

43. $9x^2 - 16y^2 - 18x - 32y - 151 = 0$

44. $-4x^2 + 25y^2 - 8x + 150y + 121 = 0$

45. $y^2 - 4x^2 - 2y - 48x + 59 = 0$

46. $9x^2 - y^2 - 72x + 8y + 119 = 0$

In Exercises 47–50, find the standard form of the equation of the hyperbola with the given characteristics.

47. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$

48. Vertices: $(0, \pm 1)$; foci: $(0, \pm 3)$

49. Foci: $(0, 0)$, $(8, 0)$; asymptotes: $y = \pm 2(x - 4)$

50. Foci: $(3, \pm 2)$; asymptotes: $y = \pm 2(x - 3)$

51. ***Navigation*** Radio transmitting station A is located 200 miles east of transmitting station B. A ship is in an area to the north and 40 miles west of station A. Synchronized radio pulses transmitted at 186,000 miles per second by the two stations are received 0.0005 second sooner from station A than from station B. How far north is the ship?

52. ***Sound Location*** Two of your friends live 4 miles apart on the same "east-west" street, and you live halfway between them. You are having a three-way phone conversation when you hear an explosion. Six seconds later your friend to the east hears the explosion, and your friend to the west hears it 8 seconds after you do. Find equations of two hyperbolas that would locate the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

In Exercises 53–56, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

53. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

54. $4x^2 + 4y^2 - 4x + 8y - 11 = 0$

55. $5x^2 - 2y^2 + 10x - 4y + 17 = 0$

56. $-4y^2 + 5x + 3y + 7 = 0$

9.4 In Exercises 57–60, rotate the axes to eliminate the *xy*-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.

57. $xy - 4 = 0$

58. $x^2 - 10xy + y^2 + 1 = 0$

59. $5x^2 - 2xy + 5y^2 - 12 = 0$

60. $4x^2 + 8xy + 4y^2 + 7\sqrt{2}x + 9\sqrt{2}y = 0$

In Exercises 61–64, (a) use the discriminant to classify the graph of the equation, (b) use the Quadratic Formula to solve for *y*, and (c) use a graphing utility to graph the equation.

61. $16x^2 - 8xy + y^2 - 10x + 5y = 0$

62. $13x^2 - 8xy + 7y^2 - 45 = 0$

63. $x^2 + y^2 + 2xy + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$

64. $x^2 - 10xy + y^2 + 1 = 0$

In Exercises 65 and 66, use any method to solve the system of quadratic equations algebraically. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

65. $\begin{cases} -4x^2 - y^2 - 32x + 24y - 64 = 0 \\ 4x^2 + y^2 + 56x - 24y + 304 = 0 \end{cases}$

66. $\begin{cases} x^2 + y^2 - 25 = 0 \\ 9x - 4y^2 = 0 \end{cases}$

9.5 In Exercises 67 and 68, complete the table for the set of parametric equations. Plot the points (x, y) and sketch a graph of the parametric equations.

67. $x = 3t - 2$ and $y = 7 - 4t$

t	-2	-1	0	1	2	3
x						
y						

68. $x = \sqrt{t}$ and $y = 8 - t$

t	0	1	2	3	4
x					
y					

In Exercises 69–74, sketch the curve represented by the parametric equations (indicate the orientation of the curve). Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.

69. $x = 5t - 1$, $y = 2t + 5$

70. $x = 4t + 1$, $y = 8 - 3t$

71. $x = t^2 + 2$, $y = 4t^2 - 3$

72. $x = \ln 4t$, $y = t^2$

73. $x = t^3$, $y = \frac{1}{2}t^2$

74. $x = \frac{4}{t}$, $y = t^2 - 1$

In Exercises 75–86, use a graphing utility to graph the curve represented by the parametric equations.

75. $x = \sqrt[3]{t}, y = t$

76. $x = t, y = \sqrt[3]{t}$

77. $x = \frac{1}{t}, y = t$

78. $x = t, y = \frac{1}{t}$

79. $x = 2t, y = 4t$

80. $x = t^2, y = \sqrt{t}$

81. $x = 1 + 4t, y = 2 - 3t$

82. $x = t + 4, y = t^2$

83. $x = 3, y = t$

84. $x = t, y = 2$

85. $x = 6 \cos \theta$, $y = 6 \sin \theta$

86. $x = 3 + 3 \cos \theta$, $y = 2 + 5 \sin \theta$

In Exercises 87–90, find two different sets of parametric equations for the given rectangular equation. (There are many correct answers.)

87. $y = 6x + 2$

88. $y = 10 - x$

89. $y = x^2 + 2$

90. $y = 2x^3 + 5x$

In Exercises 91–94, find a set of parametric equations for the line that passes through the given points. (There are many correct answers.)

91. $(3, 5), (8, 5)$

92. $(2, -1), (2, 4)$

93. $(-1, 6), (10, 0)$

94. $(0, 0), \left(\frac{5}{2}, 6\right)$

Sports **In Exercises 95–98, the quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal. The parametric equations for the path of the football are given by $x = 0.82v_0t$ and $y = 7 + 0.57v_0t - 16t^2$ where v_0 is the speed of the football (in feet per second) when it is released.**

95. Find the speed of the football when it is released.

96. Write a set of parametric equations for the path of the ball.

97. Use a graphing utility to graph the path of the ball and approximate its maximum height.

98. Find the time the receiver has to position himself after the quarterback releases the ball.

9.6 In Exercises 99–104, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.

99. $\left(1, \frac{\pi}{4}\right)$

100. $\left(-5, -\frac{\pi}{3}\right)$

101. $\left(-2, -\frac{11\pi}{6}\right)$

102. $\left(1, \frac{5\pi}{6}\right)$

103. $\left(\sqrt{5}, -\frac{4\pi}{3}\right)$

104. $\left(\sqrt{10}, \frac{3\pi}{4}\right)$

In Exercises 105–110, plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

105. $\left(5, -\frac{7\pi}{6}\right)$

106. $\left(-4, \frac{2\pi}{3}\right)$

107. $\left(2, -\frac{5\pi}{3}\right)$

108. $\left(-1, \frac{11\pi}{6}\right)$

109. $\left(3, \frac{3\pi}{4}\right)$

110. $\left(0, \frac{\pi}{2}\right)$

In Exercises 111–114, plot the point given in rectangular coordinates and find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

111. $(0, -9)$

112. $(-3, 4)$

113. $(5, -5)$

114. $(-3, -\sqrt{3})$

In Exercises 115–122, convert the rectangular equation to polar form.

115. $x^2 + y^2 = 9$

116. $x^2 + y^2 = 20$

117. $x^2 + y^2 - 4x = 0$

118. $x^2 + y^2 - 6y = 0$

119. $xy = 5$

120. $xy = -2$

121. $4x^2 + y^2 = 1$

122. $2x^2 + 3y^2 = 1$

In Exercises 123–130, convert the polar equation to rectangular form.

123. $r = 5$

124. $r = 12$

125. $r = 3 \cos \theta$

126. $r = 8 \sin \theta$

127. $r^2 = \cos 2\theta$

128. $r^2 = \sin \theta$

129. $\theta = \dfrac{5\pi}{6}$

130. $\theta = \dfrac{4\pi}{3}$

9.7 In Exercises 131–136, sketch the graph of the polar equation by hand. Then use a graphing utility to verify your graph.

131. $r = 5$

132. $r = 3$

133. $\theta = \dfrac{\pi}{2}$

134. $\theta = -\dfrac{5\pi}{6}$

135. $r = 5 \cos \theta$

136. $r = 2 \sin \theta$

In Exercises 137–144, identify and then sketch the graph of the polar equation. Identify any symmetry, maximum *r*-values, and zeros of *r*. Use a graphing utility to verify your graph.

137. $r = 5 + 4 \cos \theta$

138. $r = 1 + 4 \sin \theta$

139. $r = 3 - 5 \sin \theta$

140. $r = 2 - 6 \cos \theta$

141. $r = -3 \cos 2\theta$

142. $r = \cos 5\theta$

143. $r^2 = 5 \sin 2\theta$

144. $r^2 = \cos 2\theta$

9.8 In Exercises 145–150, identify the conic represented by the equation algebraically. Then use a graphing utility to graph the polar equation.

145. $r = \dfrac{2}{1 - \sin \theta}$

146. $r = \dfrac{1}{1 + 2 \sin \theta}$

147. $r = \dfrac{4}{5 - 3 \cos \theta}$

148. $r = \dfrac{6}{-1 + 4 \cos \theta}$

149. $r = \dfrac{5}{6 + 2 \sin \theta}$

150. $r = \dfrac{3}{4 - 4 \cos \theta}$

In Exercises 151–154, find a polar equation of the conic with its focus at the pole.

151. Parabola, vertex: $(2, \pi)$

152. Parabola, vertex: $(2, \pi/2)$

153. Ellipse, vertices: $(5, 0)$, $(1, \pi)$

154. Hyperbola, vertices: $(1, 0)$, $(7, 0)$

155. ***Planetary Motion*** The planet Mars has an elliptical orbit with an eccentricity of $e \approx 0.093$. The length of the major axis of the orbit is approximately 3.05 astronomical units. Find a polar equation for the orbit and its perihelion and aphelion distances.

156. ***Astronomy*** An asteroid takes a parabolic path with Earth as its focus. It is about 6,000,000 miles from Earth at its closest approach. Write the polar equation of the path of the asteroid with its vertex at $\theta = -\pi/2$. Find the distance between the asteroid and Earth when $\theta = -\pi/3$.

Synthesis

True or False? **In Exercises 157 and 158, determine whether the statement is true or false. Justify your answer.**

157. The graph of $\frac{1}{4}x^2 - y^4 = 1$ represents the equation of a hyperbola.

158. There is only one set of parametric equations that represents the line $y = 3 - 2x$.

Writing **In Exercises 159 and 160, an equation and four variations are given. In your own words, describe how the graph of each of the variations differs from the graph of the original equation.**

159. $y^2 = 8x$

(a) $(y - 2)^2 = 8x$

(b) $y^2 = 8(x + 1)$

(c) $y^2 = -8x$

(d) $y^2 = 4x$

160. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

(d) $\dfrac{(x - 3)^2}{4} + \dfrac{y^2}{9} = 1$

161. Consider an ellipse whose major axis is horizontal and 10 units in length. The number b in the standard form of the equation of the ellipse must be less than what real number? Describe the change in the shape of the ellipse as b approaches this number.

162. The graph of the parametric equations $x = 2 \sec t$ and $y = 3 \tan t$ is shown in the figure. Would the graph change for the equations $x = 2 \sec(-t)$ and $y = 3 \tan(-t)$? If so, how would it change?

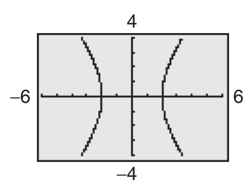


Figure for 162

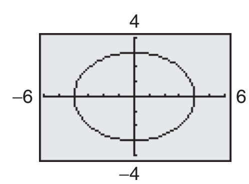


Figure for 163

163. The path of a moving object is modeled by the parametric equations $x = 4 \cos t$ and $y = 3 \sin t$, where t is time (see figure). How would the path change for each of the following?

(a) $x = 4 \cos 2t$, $y = 3 \sin 2t$

(b) $x = 5 \cos t$, $y = 3 \sin t$

9 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, graph the conic and identify any vertices and foci.

1. $y^2 - 8x = 0$ **2.** $y^2 - 4x + 4 = 0$ **3.** $x^2 - 4y^2 - 4x = 0$

4. Find the standard form of the equation of the parabola with focus $(8, -2)$ and directrix $x = 4$, and sketch the parabola.

5. Find the standard form of the equation of the ellipse shown at the right.

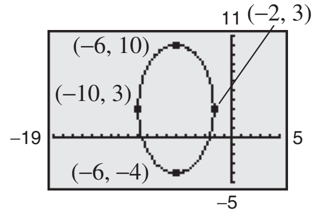


Figure for 5

6. Find the standard form of the equation of the hyperbola with vertices $(0, \pm 3)$ and asymptotes $y = \pm\frac{3}{2}x$.

7. Use a graphing utility to graph the conic $x^2 - \dfrac{y^2}{4} = 1$. Describe your viewing window.

8. (a) Determine the number of degrees the axis must be rotated to eliminate the xy-term of the conic $x^2 + 6xy + y^2 - 6 = 0$.

(b) Graph the conic in part (a) and use a graphing utility to confirm your result.

9. Solve the system of equations at the right algebraically by the method of substitution. Then verify your results by using a graphing utility.

$$\begin{cases} x^2 + 2y^2 - 4x + 6y - 5 = 0 \\ x + y + 5 = 0 \end{cases}$$

System for 9

In Exercises 10–12, sketch the curve represented by the parametric equations. Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve.

10. $x = t^2 - 6$, $y = \dfrac{1}{2}t - 1$

11. $x = \sqrt{t^2 + 2}$, $y = \dfrac{t}{4}$

12. $x = 2 + 3\cos\theta$, $y = 2\sin\theta$

In Exercises 13–15, find two different sets of parametric equations for the given rectangular equation. (There are many correct answers.)

13. $y = x^2 + 10$ **14.** $x + y^2 = 4$ **15.** $x^2 + 4y^2 - 16 = 0$

16. Convert the polar coordinate $\left(-14, \dfrac{5\pi}{3}\right)$ to rectangular form.

17. Convert the rectangular coordinate $(2, -2)$ to polar form and find two additional polar representations of this point. (There are many correct answers.)

18. Convert the rectangular equation $x^2 + y^2 - 12y = 0$ to polar form.

19. Convert the polar equation $r = 2\sin\theta$ to rectangular form.

In Exercises 20–22, identify the conic represented by the polar equation algebraically. Then use a graphing utility to graph the polar equation.

20. $r = 2 + 3\sin\theta$ **21.** $r = \dfrac{1}{1 - \cos\theta}$ **22.** $r = \dfrac{4}{2 + 3\sin\theta}$

23. Find a polar equation of an ellipse with its focus at the pole, an eccentricity of $e = \frac{1}{4}$, and directrix at $y = 4$.

24. Find a polar equation of a hyperbola with its focus at the pole, an eccentricity of $e = \frac{5}{4}$, and directrix at $y = 2$.

25. For the polar equation $r = 8\cos 3\theta$, find the maximum value of $|r|$ and any zeros of r. Verify your answers numerically.

7–9 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test to review the material in Chapters 7–9. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, use any method to solve the system of equations.

1. $\begin{cases} -x - 3y = 5 \\ 4x + 2y = 10 \end{cases}$

2. $\begin{cases} 2x - y^2 = 0 \\ x - y = 4 \end{cases}$

3. $\begin{cases} 2x - 3y + z = 13 \\ -4x + y - 2z = -6 \\ x - 3y + 3z = 12 \end{cases}$

4. $\begin{cases} x - 4y + 3z = 5 \\ 5x + 2y - z = 1 \\ -2x - 8y = 30 \end{cases}$

In Exercises 5–8, perform the matrix operations given

$$A = \begin{bmatrix} -3 & 0 & -4 \\ 2 & 4 & 5 \\ -4 & 8 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 6 & -3 & 3 \\ 0 & 4 & -2 \end{bmatrix}.$$

5. $3A - 2B$ **6.** $5A + 3B$ **7.** AB **8.** BA

9. Find (a) the inverse of A (if it exists) and (b) the determinant of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$$

10. Use a determinant to find the area of the triangle with vertices $(0, 0)$, $(6, 2)$, and $(8, 10)$.

11. Write the first five terms of each sequence a_n. (Assume that n begins with 1.)

(a) $a_n = \dfrac{(-1)^{n+1}}{2n + 3}$ (b) $a_n = 3(2)^{n-1}$

In Exercises 12–15, find the sum. Use a graphing utility to verify your result.

12. $\sum_{k=1}^{6} (7k - 2)$ **13.** $\sum_{k=1}^{4} \dfrac{2}{k^2 + 4}$ **14.** $\sum_{n=0}^{10} 9\left(\dfrac{3}{4}\right)^n$ **15.** $\sum_{n=0}^{50} 100\left(-\dfrac{1}{2}\right)^n$

In Exercises 16–18, find the sum of the inifinite geometric series.

16. $\sum_{n=0}^{\infty} 3\left(-\dfrac{3}{5}\right)^n$ **17.** $\sum_{n=1}^{\infty} 5(-0.02)^n$ **18.** $4 - 2 + 1 - \dfrac{1}{2} + \dfrac{1}{4} - \cdots$

19. Use mathematical induction to prove the formula

$$3 + 7 + 11 + 15 + \cdots + (4n - 1) = n(2n + 1).$$

In Exercises 20–23, use the Binomial Theorem to expand and simplify the expression.

20. $(x + 3)^4$ **21.** $(2x + y^2)^5$ **22.** $(x - 2y)^6$ **23.** $(3a - 4b)^8$

In Exercises 24–27, find the number of distinguishable permutations of the group of letters.

24. M, I, A, M, I

25. B, U, B, B, L, E

26. B, A, S, K, E, T, B, A, L, L

27. A, N, T, A, R, C, T, I, C, A

In Exercises 28–31, identify the conic and sketch its graph.

28. $\dfrac{(y + 3)^2}{36} - \dfrac{(x - 5)^2}{121} = 1$

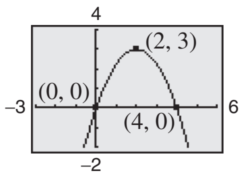
29. $\dfrac{(x - 2)^2}{4} + \dfrac{(y + 1)^2}{9} = 1$

30. $y^2 - x^2 = 16$

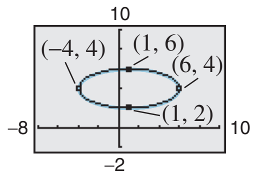
31. $x^2 + y^2 - 2x - 4y + 1 = 0$

In Exercises 32–34, find the standard form of the equation of the conic.

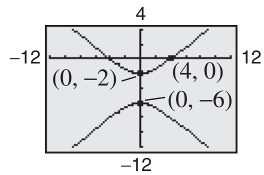
32.



33.



34.



35. Use a graphing utility to graph $x^2 - 4xy + 2y^2 = 6$. Determine the angle θ through which the axes are rotated.

In Exercises 36–38, (a) sketch the curve represented by the parametric equations, (b) use a graphing utility to verify your graph, and (c) eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.

36. $x = 2t + 1$
$y = t^2$

37. $x = \cos\theta$
$y = 2\sin^2\theta$

38. $x = 4\ln t$
$y = \frac{1}{2}t^2$

In Exercises 39–42, find two different sets of parametric equations for the given rectangular equation. (There are many correct answers.)

39. $y = 3x - 2$

40. $x^2 - y^2 = 16$

41. $y = \dfrac{2}{x}$

42. $y = \dfrac{e^{2x}}{e^{2x} + 1}$

In Exercises 43–46, plot the point given in polar coordinates and find three additional polar representations for $-2\pi < \theta < 2\pi$.

43. $\left(8, \dfrac{5\pi}{6}\right)$

44. $\left(5, -\dfrac{3\pi}{4}\right)$

45. $\left(-2, \dfrac{5\pi}{4}\right)$

46. $\left(-3, -\dfrac{11\pi}{6}\right)$

47. Convert the rectangular equation $4x + 4y + 1 = 0$ to polar form.

48. Convert the polar equation $r = 2\cos\theta$ to rectangular form.

49. Convert the polar equation $r = \dfrac{2}{4 - 5\cos\theta}$ to rectangular form.

In Exercises 50–52, identify the graph represented by the polar equation algebraically. Then use a graphing utility to graph the polar equation.

50. $r = -\dfrac{\pi}{6}$

51. $r = 3 - 2\sin\theta$

52. $r = 2 + 5\cos\theta$

53. The salary for the first year of a job is \$32,500. During the next 14 years, the salary increases by 5% each year. Determine the total compensation over the 15-year period.

54. On a game show, the digits 3, 4, and 5 must be arranged in the proper order to form the price of an appliance. If they are arranged correctly, the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$400?

55. A parabolic archway is 16 meters high at the vertex. At a height of 14 meters, the width of the archway is 12 meters, as shown in the figure at the right. How wide is the archway at ground level?

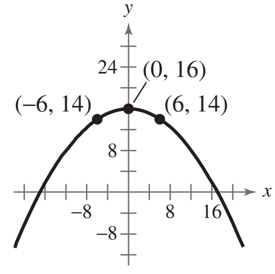


Figure for 55

Proofs in Mathematics

Standard Equation of a Parabola (p. 663)

The standard form of the equation of a parabola with vertex at (h, k) is as follows.

$(x - h)^2 = 4p(y - k), \quad p \neq 0$ Vertical axis, directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \quad p \neq 0$ Horizontal axis, directrix: $x = h - p$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.

$x^2 = 4py$ Vertical axis

$y^2 = 4px$ Horizontal axis

Parabolic Paths

There are many natural occurrences of parabolas in real life. For instance, the famous astronomer Galileo discovered in the 17th century that an object that is projected upward and obliquely to the pull of gravity travels in a parabolic path. Examples of this are the center of gravity of a jumping dolphin and the path of water molecules in a drinking fountain.

Proof

For the case in which the directrix is parallel to the x-axis and the focus lies above the vertex, as shown in the top figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus $(h, k + p)$ and the directrix $y = k - p$. So, you have

$$\sqrt{(x-h)^2 + [y-(k+p)]^2} = y - (k-p)$$

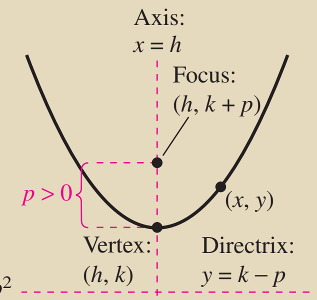
$$(x-h)^2 + [y-(k+p)]^2 = [y-(k-p)]^2$$

$$(x-h)^2 + y^2 - 2y(k+p) + (k+p)^2 = y^2 - 2y(k-p) + (k-p)^2$$

$$(x-h)^2 + y^2 - 2ky - 2py + k^2 + 2pk + p^2 = y^2 - 2ky + 2py + k^2 - 2pk + p^2$$

$$(x-h)^2 - 2py + 2pk = 2py - 2pk$$

$$(x-h)^2 = 4p(y-k).$$



Parabola with vertical axis

For the case in which the directrix is parallel to the y-axis and the focus lies to the right of the vertex, as shown in the bottom figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus $(h + p, k)$ and the directrix $x = h - p$. So, you have

$$\sqrt{[x-(h+p)]^2 + (y-k)^2} = x - (h-p)$$

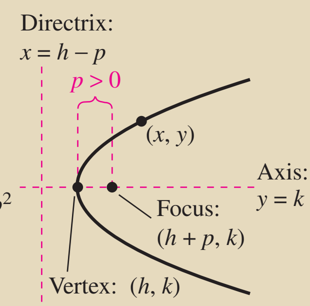
$$[x-(h+p)]^2 + (y-k)^2 = [x-(h-p)]^2$$

$$x^2 - 2x(h+p) + (h+p)^2 + (y-k)^2 = x^2 - 2x(h-p) + (h-p)^2$$

$$x^2 - 2hx - 2px + h^2 + 2ph + p^2 + (y-k)^2 = x^2 - 2hx + 2px + h^2 - 2ph + p^2$$

$$-2px + 2ph + (y-k)^2 = 2px - 2ph$$

$$(y-k)^2 = 4p(x-h).$$



Parabola with horizontal axis

Note that if a parabola is centered at the origin, then the two equations above would simplify to $x^2 = 4py$ and $y^2 = 4px$, respectively.

Rotation of Axes to Eliminate an xy-Term (p. 690)

The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

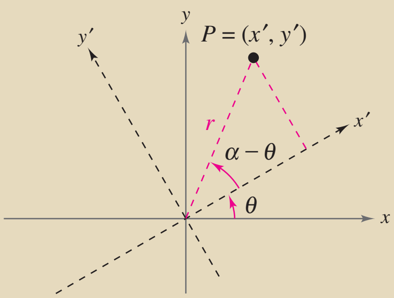
by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

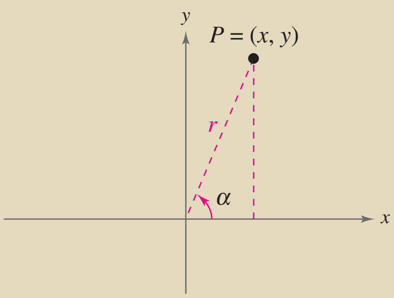
The coefficients of the new equation are obtained by making the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Proof

You need to discover how the coordinates in the xy-system are related to the coordinates in the $x'y'$-system. To do this, choose a point $P = (x, y)$ in the original system and attempt to find its coordinates (x', y') in the rotated system. In either system, the distance r between the point P and the origin is the same. So, the equations for x, y, x', and y' are those given in the figure. Using the formulas for the sine and cosine of the difference of two angles, you have the following.



Rotated: $x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$



Original: $x = r\cos\alpha$
$y = r\sin\alpha$

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ &= x\cos\theta + y\sin\theta \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\alpha - \theta) \\ &= r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \\ &= y\cos\theta - x\sin\theta \end{aligned}$$

Solving this system for x and y yields

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

Finally, by substituting these values for x and y into the original equation and collecting terms, you obtain

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ C' &= A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F. \end{aligned}$$

To eliminate the $x'y'$-term, you must select θ such that $B' = 0$.

$$\begin{aligned} B' &= 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ &= (C - A)\sin 2\theta + B\cos 2\theta \\ &= B(\sin 2\theta)\left(\frac{C - A}{B} + \cot 2\theta\right) = 0, \quad \sin 2\theta \neq 0 \end{aligned}$$

If $B = 0$, no rotation is necessary because the xy-term is not present in the original equation. If $B \neq 0$, the only way to make $B' = 0$ is to let

$$\cot 2\theta = \frac{A - C}{B}, \quad B \neq 0.$$

So, you have established the desired results.

Polar Equations of Conics (p. 722)

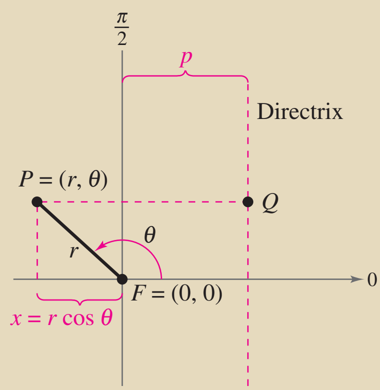
The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e \cos \theta}$ or **2.** $r = \dfrac{ep}{1 \pm e \sin \theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Proof

A proof for $r = ep/(1 + e \cos \theta)$ with $p > 0$ is shown here. The proofs of the other cases are similar. In the figure, consider a vertical directrix, p units to the right of the focus $F = (0, 0)$. If $P = (r, \theta)$ is a point on the graph of



$$r = \frac{ep}{1 + e \cos \theta}$$

the distance between P and the directrix is

$$\begin{aligned} PQ &= |p - x| \\ &= |p - r \cos \theta| \\ &= \left| p - \left(\frac{ep}{1 + e \cos \theta} \right) \cos \theta \right| \\ &= \left| p \left(1 - \frac{e \cos \theta}{1 + e \cos \theta} \right) \right| \\ &= \left| \frac{p}{1 + e \cos \theta} \right| \\ &= \left| \frac{r}{e} \right|. \end{aligned}$$

Moreover, because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is

$$\frac{PF}{PQ} = \frac{|r|}{\left| \frac{r}{e} \right|} = |e| = e$$

and, by definition, the graph of the equation must be a conic.

Progressive Summary (Chapters 3–9)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, 9, and 11. In each progressive summary, new topics encountered for the first time appear in red.

Transcendental Functions	Systems and Series	Other Topics
Exponential, Logarithmic, Trigonometric, Inverse Trigonometric	**Systems, Sequences, Series**	**Conics, Parametric and Polar Equations,**
Rewriting Exponential form ↔ Logarithmic form Condense/expand logarithmic expressions Simplify trigonometric expressions Prove trigonometric identities Use conversion formulas Operations with vectors Powers and roots of complex numbers	**Rewriting** Row operations for systems of equations Partial fraction decomposition Operations with matrices Matrix form of a system of equations *n*th term of a sequence Summation form of a series	**Rewriting** Standard forms of conics Eliminate parameters Rectangular form ↔ Parametric form Rectangular form ↔ Polar form
Solving ***Equation*** — ***Strategy*** Exponential . . . . Take logarithm of each side Logarithmic . . . . Exponentiate each side Trigonometric . . . . . Isolate function factor, use inverse function Multiple angle or high powers . . . . . Use trigonometric identities	**Solving** ***Equation*** — ***Strategy*** System of linear equations . . . . Substitution Elimination Gaussian Gauss-Jordan Inverse matrices Cramer's Rule	**Solving** ***Equation*** — ***Strategy*** Conics . . . . Convert to standard form Convert to polar form
Analyzing ***Graphically*** Intercepts Asymptotes Minimum values Maximum values ***Algebraically*** Domain, Range Transformations Composition Inverse properties Amplitude, period Reference angles ***Numerically*** Table of values	**Analyzing** Systems: Intersecting, parallel, and coincident lines, determinants Sequences: Graphing utility in *dot* mode, *n*th term, partial sums, summation formulas	**Analyzing** Conics: Table of values, vertices, foci, axes, symmetry, asymptotes, translations, eccentricity Parametric forms: Point plotting, eliminate parameters Polar forms: Point plotting, special equations, symmetry, zeros, eccentricity, maximum *r*-values, directrix